# Intuitionistic Gödel-Löb logic, à la Simpson: labelled systems and birelational semantics

**Anupam Das** ✉
University of Birmingham, United Kingdom

**Iris van der Giessen** ✉
University of Birmingham, United Kingdom

**Sonia Marin** ✉
University of Birmingham, United Kingdom

**Abstract**

We derive an intuitionistic version of Gödel-Löb modal logic (GL) in the style of Simpson, via proof theoretic techniques. We recover a labelled system, $\ell$IGL, by restricting a non-wellfounded labelled system for GL to have only one formula on the right. The latter is obtained using techniques from cyclic proof theory, sidestepping the barrier that GL's usual frame condition (converse well-foundedness) is not first-order definable. While existing intuitionistic versions of GL are typically defined over only the box (and not the diamond), our presentation includes both modalities.

Our main result is that $\ell$IGL coincides with a corresponding semantic condition in birelational semantics: the composition of the modal relation and the intuitionistic relation is conversely well-founded. We call the resulting logic IGL. While the soundness direction is proved using standard ideas, the completeness direction is more complex and necessitates a detour through several intermediate characterisations of IGL.

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## 1 Introduction

*Gödel-Löb logic* (GL) originates in the *provability* reading of modal logic: $\Box$ is interpreted as "it is provable that", in an arithmetical theory with suitable coding capacity, inducing its corresponding *provability logic*. Löb formulated a set of necessary conditions on the provability logic of Peano Arithmetic (PA), giving rise to GL, extending basic modal logic K by what we now call *Löb's axiom*: $\Box(\Box A \to A) \to \Box A$. Somewhat astoundingly GL turns out to be *complete* for PA's provability logic, a celebrated result of Solovay [36]: all that PA can prove about its own provability is already a consequence of a relatively simple (and indeed decidable) propositional modal logic.

Proof theoretically Löb's axiom represents a form of *induction*. Indeed, at the level of modal logic semantics, GL enjoys a correspondence with transitive relational structures that are *terminating*[1] [32]. Duly, in computer science, Löb's axiom has inspired several

[1] Other authors refer to this property as *Noetherian* or *conversely well-founded*, but we opt for this simpler nomenclature.

variants of modal type theories that extend simply-typed lambda calculus with some form of *recursion*. These range, for instance, from the seminal work of Nakano [26], to more recent explorations into guarded recursion [4] and intensional recursion [21]. Based in type theories, these developments naturally cast $\mathsf{GL}$ in a *constructive* setting, but little attention was given to analysing the induced intuitionistic modal logics. Indeed, for this reason, [8] has proposed a more foundational basis for studying computational interpretations of $\mathsf{GL}$, by way of a sequent calculus for the logic of the topos of trees.

Returning to the provability reading of modal logic, several constructive variants of $\mathsf{GL}$ have been independently proposed (see [23, 15] for overviews). An important such logic is $\mathsf{iGL}$ (and its variants) which now enjoys a rich mathematical theory, from semantics (e.g., [23]) to proof theory (e.g., [16, 18]). Interestingly, while $\mathsf{iGL}$ is known to be sound for the provability logic of Heyting Arithmetic (HA), the intuitionistic counterpart of PA, it is not complete. The provability logic of HA has been recently announced by [25], currently under review.

One shortfall of all the above mentioned approaches is that they do not allow us to recover a bona fide computational interpretation of *classical* $\mathsf{GL}$ along, say, the Gödel-Gentzen negative translation (GG), a standard way of lifting interpretations of intuitionistic logics to their classical counterparts. Indeed it was recently observed that the 'i' (or 'constructive') traditions of intuitionistic modal logic are too weak to validate the GG translation [10, 11].

On the other hand modal logic's relational semantics effectively renders it a fragment of usual first-order predicate logic (FOL), the so-called *standard translation*. Interpreting this semantics in an intuitionistic meta-theory defines a logic that *does* validate GG, for the same reason that intuitionistic FOL GG-interprets classical FOL. This is the approach taken (and considerably developed) by Simpson [34], building on earlier work of Fischer Servi [14] and Plotkin and Stirling [29]. The resulting logic $\mathsf{IK}$ (and friends) enjoys a remarkably robust proof theory by way of *labelled systems*, which may be duly seen as a fragment of Gentzen's systems for intuitionistic FOL by way of the standard translation.

## Contribution

In this work we develop an intuitionistic version of $\mathsf{GL}$, following Simpson's methodology [34]. In particular, while logics such as $\mathsf{iGL}$ are defined over only the $\Box$, our logic is naturally formulated over both the $\Box$ and the $\Diamond$. A key stumbling block to this end, as noticed already in the classical setting by Negri in [27], is that $\mathsf{GL}$'s correspondence to terminating relations cannot seemingly be inlined within a standard labelled system: termination is not even FOL-definable. To side-step this barrier we draw from a now well-developed proof-theoretic approach to (co)induction: *non-wellfounded proofs* (e.g. [28, 7, 35, 2, 12, 9]). Such proofs allow infinite branches, and so are (typically) equipped with a *progress condition* that ensures sound reasoning. Starting from a standard labelled system for transitive relations, $\mathsf{K4}$, we recover a labelled calculus $\ell\mathsf{GL}$ for $\mathsf{GL}$ by allowing non-wellfounded derivations under a typical progress condition. In fact, our progress condition is precisely the one from Simpson's *Cyclic Arithmetic* [35, 9]. Following (the same) Simpson, we duly recover an intuitionistic version $\ell\mathsf{IGL}$ of $\mathsf{GL}$ *syntactically* in a standard way: we restrict $\ell\mathsf{GL}$ to one formula on the right. Note that this presentation, albeit syntactic, does not constitute an *axiomatisation*, as the system $\ell\mathsf{IGL}$ is infinitary.

At the same time we can recover an intuitionistic version of $\mathsf{GL}$ *semantically* by suitably adapting the *birelational semantics* of intuitionistic modal logics, which combine the partial order $\leq$ of intuitionistic semantics with the accessibility relation $R$ of modal semantics. Our semantic formulation $\mathscr{B}\mathsf{IGL}$ is duly obtained by reading the termination criterion of classical $\mathsf{GL}$'s semantics intuitionistically: the composition $\leq; R$ must be terminating. Our

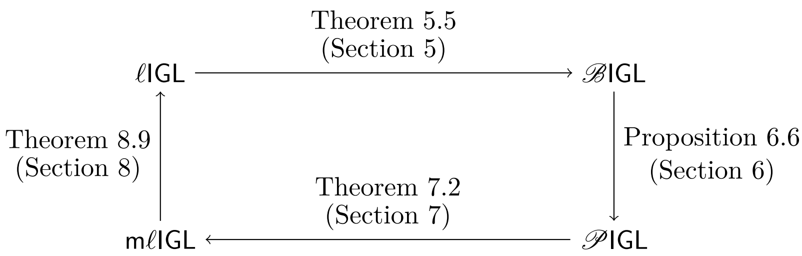



**Figure 1** Summary of our main results, where arrows $\to$ denote inclusions of modal logics.

main result is that these two characterisations, $\ell$IGL and $\mathscr{B}$IGL, are indeed equivalent, and we duly dub the resulting logic IGL. Note that we do not address any *provability* reading of IGL in this work, being beyond scope.

The soundness direction is proved via standard techniques from intuitionistic modal logic and non-wellfounded proof theory. The completeness direction, on the other hand, is more cumbersome. To this end we exercise an intricate combination of proof theoretic techniques, necessitating two further (and ultimately equivalent) characterisations of IGL: semantically $\mathscr{P}$IGL, essentially a class of intuitionistic FOL structures, and syntactically m$\ell$IGL, a *multi-succedent* variant of $\ell$IGL. These formulations facilitate a countermodel construction from failed proof search, inspired by Takeuti's for LJ [37]. Due to the non-wellfoundedness of proofs we employ a determinacy principle to organise the construction, a standard technique in non-wellfounded and cyclic proof theory (e.g. [28]). For the reduction to $\ell$IGL we devise a form of *continuous cut-elimination*, building on more recent ideas in non-wellfounded proof theory, cf. [2, 12]. Our 'grand tour' of results is visualised in Figure 1, also indicating the organisation of this paper. Due to space constraints full proofs are omitted, but may be found in an associated preprint [13].

### Other related work

The proof theory of GL is (in)famously complex. The first sequent calculus of GL was considered in [22] but its cut-elimination property was only finally settled (positively) in [17] after several attempts [38, 31, 24]. Intuitionistic versions of sequent calculi for GL are developed in [16, 18] and provide calculi for iGL. Labelled calculi [27] and *nested calculi* [30] have also been proposed. However all of these calculi are arguably unsatisfactory: the modality introduction rules are non-standard (which is atypical for labelled calculi) and, in particular, modal rules may change the polarity of formula occurrences, due to the inductive nature of Löb's axiom.

It is possible to recover a system for GL by admitting non-wellfounded proofs in a sequent calculus for K4, as observed by Shamkanov [33]. An intuitionistic version of this system has been studied by Iemhoff [20]. In these works both the base calculus and the corresponding correctness criterion are bespoke, rather than 'recovered' from established foundations.

Non-wellfounded proofs originate in the study of modal logics with fixed points, in particular Niwinski and Walukiewicz's seminal work on the $\mu$-calculus [28]. These ideas were later inlined into the proof theory of FOL with certain inductive definitions by Brotherston and Simpson (e.g. [7]), a source of inspiration for the present work. As already mentioned, recent extensions of these ideas to PA [35, 9] and advances on cut-elimination [2, 12] are quite relevant to our development.

# 2 Preliminaries on (classical) modal logic

Throughout this work we work with a set $\mathsf{Pr}$ of **propositional symbols**, written $p, q$, etc., which we simultaneously construe as unary predicate symbols when working in predicate logic. For the latter we also assume a single binary relation symbol $R$ and a countable set $\mathsf{Var}$ of **(individual) variables**, written $x, y$, etc.

We recall some preliminaries on classical modal logic in this section, but point the reader to general references such as [5, 6] for a more comprehensive background.

## 2.1 Language and semantics

The formulas of modal logic are generated by the following grammar:

$$A \quad ::= \quad p \in \mathsf{Pr} \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A \mid \Diamond A$$

As usual we write $\neg A := A \to \bot$, and employ standard bracketing conventions.

Modal formulas are interpreted over **relational frames** $\mathcal{F} = (W, R^{\mathcal{F}})$ formed of a non empty set of **worlds** $W$ equipped with an **accessibility relation** $R^{\mathcal{F}} \subseteq W \times W$.[2] A **relational model** $\mathcal{M} = (W, R^{\mathcal{M}}, V)$ is a structure where $(W, R^{\mathcal{M}})$ is a frame with a **valuation** $V : W \to \mathcal{P}(\mathsf{Pr})$. Let $\mathcal{M} = (W, R^{\mathcal{M}}, V)$ be a relational model. For worlds $w \in W$ and formulas $A$ we define the satisfaction judgement $\mathcal{M}, w \vDash A$ as follows:

- $\mathcal{M}, w \vDash p$ if $p \in V(w)$;
- $\mathcal{M}, w \nvDash \bot$;
- $\mathcal{M}, w \vDash A \wedge B$ if $\mathcal{M}, w \vDash A$ and $\mathcal{M}, w \vDash B$;
- $\mathcal{M}, w \vDash A \vee B$ if $\mathcal{M}, w \vDash A$ or $\mathcal{M}, w \vDash B$;
- $\mathcal{M}, w \vDash A \to B$ if $\mathcal{M}, w \vDash A$ implies $\mathcal{M}, w \vDash B$;
- $\mathcal{M}, w \vDash \Box A$ if for all $v$ such that $wR^{\mathcal{M}}v$ we have $\mathcal{M}, v \vDash A$;
- $\mathcal{M}, w \vDash \Diamond A$ if there exists $v$ such that $wR^{\mathcal{M}}v$ and $\mathcal{M}, v \vDash A$.

If $\mathcal{M}, w \vDash A$ for all $w \in W$ we simply write $\mathcal{M} \vDash A$, and if $(W, R^{\mathcal{M}}, V) \vDash A$ for all valuations $V$ based on frame $\mathcal{F} = (W, R^{\mathcal{M}})$, we simply write $\mathcal{F} \vDash A$.

## 2.2 Axiomatisations and Gödel-Löb logic

Turning to syntax, let us now build up the modal logics we are concerned with axiomatically, before relating them to the semantics just discussed. The modal logic $\mathsf{K}$ is axiomatised by all the theorems of classical propositional logic (CPL) together with axiom $(\mathsf{k})$ and closed under the rules $(\mathsf{mp})$ (*modus ponens*) and $\mathsf{nec}$ (*necessitation*) from Figure 2. The logic $\mathsf{K4}$ is defined in the same way by further including the axiom $(4)$ from Figure 2.

Referring back to the earlier semantics, the following characterisation is well-known [5]:

▶ **Theorem 2.1** ($\mathsf{K}$/$\mathsf{K4}$ characterisation)**.** *$\mathsf{K} \vdash A$ (resp. $\mathsf{K4} \vdash A$) iff $A$ is satisfied in all relational frames (resp. with transitive accessibility relation).*

The main subject of this work is an extension of $\mathsf{K}$ by the Löb axiom $(\mathsf{löb})$, in Figure 2, a sort of induction principle:

[2] We parameterise the relations by the frame or models to be able distinguish it from the fixed binary relation symbol $R$.

$$(\mathsf{k}) : \Box(A \to B) \to (\Box A \to \Box B) \qquad (\mathsf{mp})\ \frac{A \to B \quad A}{B} \qquad (\mathsf{nec})\ \frac{A}{\Box A}$$

$$(4) : \Box A \to \Box\Box A \qquad (\text{löb}) : \Box(\Box A \to A) \to \Box A$$

**Figure 2** Some modal axioms and rules.

▶ **Definition 2.2** (Gödel-Löb logic). GL *is defined by extending* K4 *by the* ***Löb axiom*** (löb) *and is closed under* (mp) *and* (nec).[3]

Semantically (löb) says that, as long as worlds satisfy $A$ whenever all its successors (wrt. the accessibility relation) satisfy $A$, then $A$ holds universally. This amounts to a 'reverse' induction principle for the accessibility relation. Indeed we have an associated characterisation for GL just like that for K4.

We call a frame **terminating** if its accessibility relation has no infinite path.

▶ **Theorem 2.3** (GL characterisation). GL $\vdash A$ *iff all transitive terminating frames satisfy* $A$.

## 2.3 Labelled calculi and the standard translation

The relational semantics of modal logic may be viewed as a bona fide fragment of predicate logic. Recalling the predicate language we fixed at the start of the section, the **standard translation** is defined as follows: for individual variables $x$ and modal formulas $A$ we define the predicate formula $x : A$ by:

- $x : p$ is $p(x)$;
- $x : \bot$ is $\bot$;
- $x : A \star B$ is $(x : A) \star (x : B)$ for $\star \in \{\vee, \wedge, \to\}$;
- $x : \Diamond A$ is $\exists y(xRy \wedge (y : A))$ ;
- $x : \Box A$ is $\forall y(xRy \to (y : A))$.

This induces a well-behaved proof theory as a fragment of usual first-order predicate systems [27], adaptable to many extensions under correspondence theorems such as Theorem 2.1.

A **(labelled) sequent** is an expression $\mathbf{R}, \Gamma \Rightarrow \Delta$, where $\mathbf{R}$ is a set of **relational atoms**, i.e. formulas of form $xRy$, and $\Gamma$ and $\Delta$ are multisets of **labelled formulas**, i.e. formulas of form $x : A$. We sometimes refer to the variable $x$ in $x : A$ as a **label**. We write $\mathsf{Var}(S)$ for the subset of labels/variables that occur in a given sequent $S$. Similarly so for $\mathsf{Var}(\mathbf{R})$, etc.

Notationally, we have identified labelled formulas with the standard translation at the beginning of this section. This is entirely suggestive, as we can now easily distil systems for modal logics of interest by appealing to the sequent calculus LK for first-order predicate logic. In this vein, the labelled calculus $\ell$K for modal logic K is given in Figure 3. From here, under the aforementioned correspondence between K4 and transitive frames, we have a system $\ell$K4 for K4 that extends $\ell$K by the **transitivity rule**:

$$\mathsf{tr}\ \frac{\mathbf{R}, xRy, yRz, xRz, \Gamma \Rightarrow \Delta}{\mathbf{R}, xRy, yRz, \Gamma \Rightarrow \Delta}$$

[3] Alternatively, GL can be axiomatised by adding the (löb) axiom to K, as (4) can be derived from (löb).

*Identity and cut:*

$$\mathsf{id}\ \frac{}{\mathbf{R}, x : p \Rightarrow x : p} \qquad \mathsf{cut}\ \frac{\mathbf{R}, \Gamma \Rightarrow \Delta, x : A \quad \mathbf{R}, \Gamma', x : A \Rightarrow \Delta'}{\mathbf{R}, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

*Structural rules:*

$$\mathsf{w}\text{-}l\ \frac{\mathbf{R}, \Gamma \Rightarrow \Delta}{\mathbf{R}, \Gamma, x : A \Rightarrow \Delta} \qquad \mathsf{c}\text{-}l\ \frac{\mathbf{R}, \Gamma, x : A, x : A \Rightarrow \Delta}{\mathbf{R}, \Gamma, x : A \Rightarrow \Delta}$$

$$\mathsf{w}\text{-}r\ \frac{\mathbf{R}, \Gamma \Rightarrow \Delta}{\mathbf{R}, \Gamma \Rightarrow \Delta, x : A} \qquad \mathsf{c}\text{-}r\ \frac{\mathbf{R}, \Gamma \Rightarrow \Delta, x : A, x : A}{\mathbf{R}, \Gamma \Rightarrow \Delta, x : A}$$

*Relational structural rule:*

$$\mathsf{th}\ \frac{\mathbf{R}, \Gamma \Rightarrow \Delta}{\mathbf{R}, \mathbf{R}', \Gamma \Rightarrow \Delta}$$

*Propositional logical rules:*

$$\bot\text{-}l\ \frac{}{\mathbf{R}, x : \bot, \Gamma \Rightarrow \Delta} \qquad \text{(no right rule for } \bot)$$

$$\to\text{-}l\ \frac{\mathbf{R}, \Gamma \Rightarrow \Delta, x : A \quad \mathbf{R}, \Gamma', x : B \Rightarrow \Delta'}{\mathbf{R}, \Gamma, \Gamma', x : A \to B \Rightarrow \Delta, \Delta'} \qquad \to\text{-}r\ \frac{\mathbf{R}, \Gamma, x : A \Rightarrow \Delta, x : B}{\mathbf{R}, \Gamma \Rightarrow \Delta, x : A \to B}$$

$$\wedge\text{-}l\ \frac{\mathbf{R}, \Gamma, x : A_i \Rightarrow \Delta}{\mathbf{R}, \Gamma, x : A_0 \wedge A_1 \Rightarrow \Delta}\ i \in \{0,1\} \qquad \vee\text{-}l\ \frac{\mathbf{R}, \Gamma, x : A \Rightarrow \Delta \quad \mathbf{R}, \Gamma, x : B \Rightarrow \Delta}{\mathbf{R}, \Gamma, x : A \vee B \Rightarrow \Delta}$$

$$\vee\text{-}r\ \frac{\mathbf{R}, \Gamma \Rightarrow \Delta, x : A_i}{\mathbf{R}, \Gamma \Rightarrow \Delta, x : A_0 \vee A_1}\ i \in \{0,1\} \qquad \wedge\text{-}r\ \frac{\mathbf{R}, \Gamma \Rightarrow \Delta, x : A \quad \mathbf{R}, \Gamma \Rightarrow \Delta, x : B}{\mathbf{R}, \Gamma \Rightarrow \Delta, x : A \wedge B}$$

*Modal logical rules:*

$$\Diamond\text{-}l\ \frac{\mathbf{R}, xRy, \Gamma, y : A \Rightarrow \Delta}{\mathbf{R}, \Gamma, x : \Diamond A \Rightarrow \Delta}\ y \text{ fresh} \qquad \Diamond\text{-}r\ \frac{\mathbf{R}, xRy, \Gamma \Rightarrow \Delta, y : A}{\mathbf{R}, xRy, \Gamma \Rightarrow \Delta, x : \Diamond A}$$

$$\Box\text{-}r\ \frac{\mathbf{R}, xRy, \Gamma \Rightarrow \Delta, y : A}{\mathbf{R}, \Gamma \Rightarrow \Delta, x : \Box A}\ y \text{ fresh} \qquad \Box\text{-}l\ \frac{\mathbf{R}, xRy, \Gamma, y : A \Rightarrow \Delta}{\mathbf{R}, xRy, \Gamma, x : \Box A \Rightarrow \Delta}$$

**Figure 3** The standard labelled calculus $\ell\mathsf{K}$ for modal logic $\mathsf{K}$.

For a labelled system $\mathsf{S}$ we write $\mathsf{S} \vdash \mathbf{R}, \Gamma \Rightarrow \Delta$, if there is a proof of $\mathbf{R}, \Gamma \Rightarrow \Delta$ using the rules from $\mathsf{S}$. We write $\mathsf{S} \vdash x : A$, or even $\mathsf{S} \vdash A$, to mean $\mathsf{S} \vdash \varnothing \Rightarrow x : A$. Almost immediately from Theorem 2.1 and metatheorems for predicate logic, we have:

▶ **Proposition 2.4** (Soundness and completeness)**.** *$\ell\mathsf{K} \vdash x : A$ (resp. $\ell\mathsf{K4} \vdash x : A$) iff $\mathsf{K} \vdash A$ (resp. $\mathsf{K4} \vdash A$).*

Naturally systems for many other modal logics can be readily obtained, when they correspond to simple frame properties, by adding further relational (structural) rules [27]. Indeed, let us call a labelled calculus *standard* if it does not extend $\ell\mathsf{K}$ by any new logical or (non-relational) structural rules, nor any new logical axioms. This terminology is suggestive, since it forces the left and right introduction rules for modalities to coincide with those induced by the *standard* translation. As remarked by Negri in [27], typical standard calculi cannot be complete for $\mathsf{GL}$, as termination of a relation is not even first-order definable. We shall sidestep this barrier in the next section by making use of *non-wellfounded* systems.

## 3 Recovering a proof theoretic account for GL

In another branch of the proof theory literature, structural treatments of induction and well-foundedness have been developed in the guise of *non-wellfounded* and *cyclic* proofs, e.g. [28, 7, 2, 3, 35, 9]. Here non-wellfoundedness in proofs allows for inductive reasoning, and soundness is ensured by some global correctness condition. By incorporating these ideas into modal proof theory, one can design a non-wellfounded proof system for GL [33]. In this section we recover a *standard* labelled calculus for GL, in the sense of the preceding discussion. Our presentation is based on the correctness condition for *Cyclic Arithmetic* in [35, 9].

### 3.1 A standard calculus for GL, via non-wellfounded proofs

In what follows, we consider systems S that will typically be some fragment of the system $\ell$K4 given earlier. The definitions we give apply to all 'non-wellfounded' systems of this work.

▶ **Definition 3.1** (Preproofs). *A **preproof** in a system* S *is a possibly infinite derivation generated from the rules of* S.

As preproofs may, in particular, be non-wellfounded, they may conclude fallacious theorems, so we require a correctness criterion. Appealing to the correspondence of GL over transitive terminating relations, we may directly import a 'trace' condition first used in [35].

▶ **Definition 3.2** (Traces and proofs). *Fix a preproof $P$ and an infinite branch $(S_i)_{i<\omega}$. A **trace** along $(S_i)_{i<\omega}$ is a sequence of variables $(x_i)_{i<\omega}$ such that either:*
- *$x_{i+1} = x_i$; or,*
- *$x_i R x_{i+1}$ appears in $S_i$ (then $i$ is a **progress point** of $(x_i)_{i<\omega}$).*

*A trace is **progressing** if it is not ultimately constant, i.e. if 3.2 above applies infinitely often. A branch $(S_i)_{i<\omega}$ is progressing if it has a progressing trace, and a preproof $P$ is progressing, or simply is a $\infty$**-proof**, if each infinite branch has a progressing trace.*
*We write $\mathsf{S} \vdash^{\infty} \mathbf{R}, \Gamma \Rightarrow \Delta$ if there is a $\infty$-proof in* S *of the sequent $\mathbf{R}, \Gamma \Rightarrow \Delta$.*

▶ **Example 3.3** (Löb and contra-Löb). An example of a $\infty$-proof of (löb) in $\ell$K4 is given in Figure 4, left. Here we have used bullets • to identify identical subproofs, up to the indicated renamings of variables. The preproof is indeed progressing: it has only one infinite branch, whose progress points are coloured red and trigger at each iteration of the •-loop. Notice that each sequent has only one formula on the right-hand side (RHS).

Another example of a $\infty$-proof in $\ell$K4 is the *contraposition* of (löb), given in Figure 4, right. We have merged several steps, and omitted some routine structural steps and initial sequents, a convention we shall continue to employ throughout this work. Again the preproof is indeed progressing, by the same argument as before. Notice, this time, that there are two formulas in the RHS of the premiss of •. Indeed, by basic inspection of proof search, there is no cut-free $\infty$-proof avoiding this feature.

▶ Remark 3.4 (On regularity). In non-wellfounded proof theory, special attention is often paid to the subset of *regular* preproofs, which may be written as finite (possibly cyclic) graphs, as in Example 3.3 above. Nonetheless these will play no role in the present work, as we are purely concerned with logical and proof theoretic investigations, not with effectivity.

The progress condition is invariant under expansion of relational contexts in a preproof: We may omit consideration of the thinning rule th (Figure 3) when reasoning about $\infty$-proofs:

▶ **Observation 3.5.** th *is eliminable in $\infty$-proofs of* S.

$$
\begin{array}{rc}
 & \vdots \\ \hline
\textsf{c-}l & xRz, x : \Box(\Box p \to p) \Rightarrow z : p \quad \bullet[z/y] \\ \hline
\textsf{tr} & xRy, \color{red}{yRz}, x : \Box(\Box p \to p) \Rightarrow z : p \\ \hline
\Box\text{-}r & xRy, x : \Box(\Box p \to p) \Rightarrow y : \Box p \\ \hline
\to\text{-}l & xRy, x : \Box(\Box p \to p), y : \Box p \to p \Rightarrow y : p \\ \hline
\Box\text{-}l & xRy, x : \Box(\Box p \to p), x : \Box(\Box p \to p) \Rightarrow y : p \\ \hline
\textsf{c-}l & \color{red}{xRy}, x : \Box(\Box p \to p) \Rightarrow y : p \quad \bullet \\ \hline
\Box\text{-}r & x : \Box(\Box p \to p) \Rightarrow x : \Box p \\ \hline
\to\text{-}r & \Rightarrow x : \Box(\Box p \to p) \to \Box p
\end{array}
$$

$$
\begin{array}{rc}
 & \vdots \\ \hline
\Diamond\text{-}r & xRz, z : p \Rightarrow x : \Diamond(p \wedge \Box\neg p) \quad \bullet[z/y] \\ \hline
\textsf{tr} & xRy, \color{red}{yRz}, y : p, z : p \Rightarrow x : \Diamond(p \wedge \Box\neg p) \\ \hline
\neg\text{-}r, \Box\text{-}r & xRy, y : p \Rightarrow x : \Diamond(p \wedge \Box\neg p), y : \Box\neg p \\ \hline
\wedge\text{-}r & xRy, y : p \Rightarrow x : \Diamond(p \wedge \Box\neg p), y : p \wedge \Box\neg p \\ \hline
\Diamond\text{-}r & \color{red}{xRy}, y : p \Rightarrow x : \Diamond(p \wedge \Box\neg p) \quad \bullet \\ \hline
\Diamond\text{-}l & x : \Diamond p \Rightarrow \Diamond(p \wedge \Box\neg p) \\ \hline
\to\text{-}r & \Rightarrow x : \Diamond p \to \Diamond(p \wedge \Box\neg p)
\end{array}
$$

**Figure 4** An $\infty$-proof in $\ell$K4 of Löb's axiom, left, and its contraposition, right. Identity steps on $y : p$ above the $\to$-$l$ step, left, and the $\wedge$-$r$ step, right, are omitted for space considerations.

## 3.2 Soundness and completeness

Let us now argue that our notion of $\infty$-proof for $\ell$K4 is sound and complete for GL. The most interesting part is soundness, comprising a contradiction argument by infinite descent as is common in non-wellfounded proof theory, relying on the characterisation result of Theorem 2.3:

▶ **Proposition 3.6** (Soundness). *If* $\ell\mathsf{K4} \vdash^{\infty} x : A$ *then* $\mathsf{GL} \vdash A$*.*

On the other hand, thanks to Example 3.3 and the known completeness of $\ell$K4 for K4, we can use cut-rules to derive:

▶ **Proposition 3.7** (Completeness). *If* $\mathsf{GL} \vdash A$ *then* $\ell\mathsf{K4} \vdash^{\infty} x : A$*.*

These results motivate the following notation:

▶ **Definition 3.8.** *$\ell$GL is the class of $\infty$-proofs of $\ell$K4.*

Henceforth for a class of $\infty$-proofs $\mathbf{P}$ we may write simply $\mathbf{P} \vdash S$ if $\mathbf{P}$ contains a $\infty$-proof of the sequent $S$. For instance, writing $\ell\mathsf{GL} \vdash S$ is the same as $\ell\mathsf{K4} \vdash^{\infty} S$.

# 4 Recovering intuitionistic versions of GL from syntax and semantics

In this section we propose two intuitionistic versions of GL, in the style of Simpson, respectively by consideration of the proof theory and semantics of GL discussed in the previous sections. Later sections are then devoted to proving the equivalence of these two notions.

## 4.1 An intuitionistic GL, via syntax

Following Gentzen, it is natural to define intuitionistic calculi based on their classical counterparts by restricting sequents to one formula on the RHS. However, when implementing this restriction to different starting calculi for modal logic K (based on sequents, nested sequents, labelled sequents) one ends up with different intuitionistic variants of K (see e.g. [10]). The restriction of the ordinary sequent calculus for K defines a logic which is not compatible with the standard translation in the way we described for K. On the other hand labelled calculi are well designed for this purpose.

▶ **Definition 4.1.** *The system* $\ell$IK *(resp.* $\ell$IK4*) is the restriction of* $\ell$K *(resp.* $\ell$K4*) to sequents in which exactly one formula occurs on the RHS.*

Note that the rules w-$r$ and c-$r$ cannot be used in $\ell$IK4, by the singleton restriction on the RHS of a sequent. In the style of Simpson [34], we can from here duly recover a standard intuitionistic analogue of GL, by restricting $\ell$GL to $\infty$-proofs with only singleton RHSs.

▶ **Definition 4.2.** *We write* $\ell$IGL *for the class of* $\infty$*-proofs of* $\ell$IK4*.*

▶ **Example 4.3** (Löb, revisited)**.** Recalling Example 3.3 earlier, note that the $\infty$-proof of Löb's axiom in Figure 4, left, indeed satisfies the singleton RHS restriction, and so $\ell\mathsf{IGL} \vdash (\mathsf{l\ddot{o}b})$. On the other hand its contraposition, right, does not satisfy this restriction. We will see shortly in Example 4.8 that it is indeed *not* a theorem of $\ell$IGL.

## 4.2 An intuitionistic GL, via semantics

To give our semantic version of intuitionistic GL, we must first recall models of intuitionistic modal logic. *Birelational semantics* [29, 34] include models $\mathcal{B}$ with two relations, the intuitionistic $\leq$ and the modal $R^{\mathcal{B}}$ (parameterised with $\mathcal{B}$ to distinguish from the fixed predicate symbol $R$ used in this paper), which duly gives it the capacity to model intuitionistic modal logics.

▶ **Definition 4.4** (Birelational semantics)**.** *A* ***birelational frame*** $\mathcal{F}$ *is a triple* $(W, \leq, R^{\mathcal{F}})$*, where* $W$ *is a nonempty set of* ***worlds*** *equipped with a partial order* $\leq$ *and an* ***accessibility relation*** $R^{\mathcal{F}} \subseteq W \times W$*. We require the following frame conditions:*

(F1) *If* $w \leq w'$ *and* $wR^{\mathcal{F}}v$*, then there exists* $v'$ *such that* $v \leq v'$ *and* $w'R^{\mathcal{F}}v'$*.*

(F2) *If* $wR^{\mathcal{F}}v$ *and* $v \leq v'$*, then there exists* $w'$ *such that* $w \leq w'$ *and* $w'R^{\mathcal{F}}v'$*.*

*A* ***birelational model*** *is a tuple* $(W, \leq, R^{\mathcal{B}}, V)$*, where* $(W, \leq, R^{\mathcal{B}})$ *is a birelational frame and* $V$ *is a* ***valuation*** $W \to \mathcal{P}(\mathsf{Pr})$ *that is monotone in* $\leq$*, i.e.,* $w \leq w'$ *implies* $V(w) \subseteq V(w')$*.*

*Let* $\mathcal{B} = (W, \leq, R^{\mathcal{B}}, V)$ *be a birelational model. For worlds* $w \in W$ *and formulas* $A$ *we define the satisfaction judgement* $\mathcal{B}, w \vDash A$ *as follows:*

- $\mathcal{B}, w \vDash p$ *if* $p \in V(w)$*;*
- $\mathcal{B}, w \nvDash \bot$*;*
- $\mathcal{B}, w \vDash A \wedge B$ *if* $\mathcal{B}, w \vDash A$ *and* $\mathcal{B}, w \vDash B$*;*
- $\mathcal{B}, w \vDash A \vee B$ *if* $\mathcal{B}, w \vDash A$ *or* $\mathcal{B}, w \vDash B$*;*
- $\mathcal{B}, w \vDash A \to B$ *if for all* $w' \geq w$*, if* $\mathcal{B}, w' \vDash A$ *then* $\mathcal{B}, w' \vDash B$*;*
- $\mathcal{B}, w \vDash \Box A$ *if for all* $w' \geq w$ *and for all* $v$ *such that* $w'R^{\mathcal{B}}v$ *we have* $\mathcal{B}, v \vDash A$*;*
- $\mathcal{B}, w \vDash \Diamond A$ *if there exists* $v$ *such that* $wR^{\mathcal{B}}v$ *and* $\mathcal{B}, v \vDash A$*.*

*We write* $\mathcal{B} \vDash A$ *if* $\mathcal{B}, w \vDash A$ *for all* $w \in W$*.*

Note that the clauses above are essentially determined by evaluating the standard translation of modal formulas within *intuitionistic* predicate models. This is why, say, evaluating a $\Box$ requires quantifying over all $w' \geq w$ (like $\forall$), but $\Diamond$ does not (like $\exists$).

▶ **Lemma 4.5** (Monotonicity lemma, [34])**.** *Let* $\mathcal{B} = (W, \leq, R^{\mathcal{B}}, V)$ *be a birelational model. For any formula* $A$ *and* $w, w' \in W$*, if* $w \leq w'$ *and* $\mathcal{B}, w \vDash A$*, then* $\mathcal{B}, w' \vDash A$*.*

A soundness and completeness theorem is recovered in [34], similarly to the classical case (Proposition 2.4 and Theorem 2.1).

▶ **Theorem 4.6** ([34])**.** $\ell\mathsf{IK} \vdash x : A$ *(resp.* $\ell\mathsf{IK4} \vdash x : A$*) iff* $A$ *is satisfied in all birelational models (resp. with transitive accessibility relation)*

We want to introduce a birelational counterpart of the transitive and terminating models of GL. The class of models that we will introduce here is different from the birelational semantics known for logic iGL [23, 1] which only require termination of the accessibility relation. Due to the *global* nature of $\Box$-evaluation in Definition 4.4, we here require the *composition* of $\leq$ and $R^{\mathcal{B}}$ to be terminating.

▶ **Definition 4.7.** *$\mathscr{B}$IGL is the class of birelational models $\mathcal{B} = (W, \leq, R^{\mathcal{B}}, V)$ such that:*

- *$R^{\mathcal{B}}$ is transitive; and,*
- *$(\leq; R^{\mathcal{B}})$ is terminating, i.e., there are no infinite paths $x_1 \leq y_1 R^{\mathcal{B}} x_2 \leq y_2 R^{\mathcal{B}} x_3 \ldots$.*

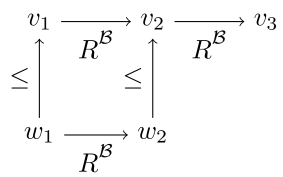
*For a formula $A$, we write $\mathscr{B}$IGL $\vDash A$ to mean that $\mathcal{B} \vDash A$ for all $\mathcal{B} \in \mathscr{B}$IGL.*

Note that the second condition, that $(\leq; R^{\mathcal{B}})$ is terminating, implies termination of $R^{\mathcal{B}}$ too, as $\leq$ is reflexive. However the converse does not in general hold. For instance if $W = (w_{ij})_{i<j<\omega}$ with:

- if $j \leq j'$ then $w_{ij} \leq w_{ij'}$;
- if $i < i'$ then $w_{ij} R^{\mathcal{B}} w_{i'j}$.

Here certainly $R^{\mathcal{B}}$ is terminating, as $R$-paths always have $j$ fixed and $i$ increasing and there are only $j$ worlds of form $w_{ij}$. On the other hand there is an infinite $(\leq; R^{\mathcal{B}})$ path: $w_{01}, w_{12}, w_{23}, \ldots$. The frame conditions of Definition 4.4 also hold.

▶ **Example 4.8** (Contra-Löb, revisited)**.** Recalling Examples 3.3 and 4.3 we indeed have that the contraposition of Löb's axiom, $\Diamond p \to \Diamond(p \wedge \Box\neg p)$, is not valid in $\mathscr{B}$IGL. It is falsified at world $w_1$ of the following $\mathscr{B}$IGL-model $\mathcal{B}$ where we assign $p$ to all worlds. In the picture we omit transitive (and reflexive) edges of $R^{\mathcal{B}}$ (and $\leq$).




## 5 Soundness

The main result of this section is the soundness of $\ell$IGL with respect to $\mathscr{B}$IGL (Theorem 5.5 and Corollary 5.6). First we must extend the notion of satisfaction in birelational models to labelled sequents.

Let us fix a sequent $S = (\mathbf{R}, \Gamma \Rightarrow x : A)$ and a birelational model $\mathcal{B} = (W, \leq, R^{\mathcal{B}}, V)$ for the remainder of this section. An **interpretation** of $S$ into $\mathcal{B}$ is a function $\mathcal{I} : \mathsf{Var}(S) \to W$ such that $\mathcal{I}(x) R^{\mathcal{B}} \mathcal{I}(y)$ whenever $xRy \in \mathbf{R}$. We write

$$\mathcal{B}, \mathcal{I} \vDash \mathbf{R}, \Gamma \Rightarrow x : A \quad \text{ if } \quad \mathcal{B}, \mathcal{I}(x) \vDash A \text{ when } \mathcal{B}, \mathcal{I}(y) \vDash B \text{ for all } y : B \in \Gamma.$$

If $\mathcal{B}, \mathcal{I} \vDash S$ for all interpretations $\mathcal{I}$ of $S$ into $\mathcal{B}$, we simply write $\mathcal{B} \vDash S$, and if $\mathcal{B} \vDash S$ for all models $\mathcal{B} \in \mathscr{B}$IGL, we simply write $\mathscr{B}$IGL $\vDash S$. See associated preprint [13] for a complete proof of the following.

▶ **Observation 5.1.** $\mathcal{B} \vDash \varnothing, x : B_1, \ldots, x : B_n \Rightarrow x : A$ *iff* $\mathcal{B} \vDash (B_1 \wedge \ldots \wedge B_n) \to A$.

In order to prove local soundness of $\ell$IK4 rules, we use a lifting lemma similarly to the one in [34] whose proof relies on the tree-like structure of $\mathbf{R}$.

▶ **Definition 5.2** ((Quasi-)tree-like). *$\mathbf{R}$ is a **tree** if there is $x_0 \in \mathsf{Var}(\mathbf{R})$ such that for each $x \in \mathsf{Var}(\mathbf{R})$, $x \neq x_0$, there is a unique sequence $x_0 R x_1, x_1 R x_2, \ldots, x_m R x \in \mathbf{R}$. $\mathbf{R}$ is a **quasi-tree** if there is some $\mathbf{R}_0 \subseteq \mathbf{R} \subseteq \mathbf{R}_0^+$ where $\mathbf{R}_0$ is a tree and $\mathbf{R}_0^+$ denotes the transitive closure of $\mathbf{R}_0$. Sequent $S$ is **(quasi-)tree-like** if either $\mathbf{R} = \varnothing$ and $\mathsf{Var}(\Gamma) \subseteq \{x\}$, or $\mathbf{R}$ is a (resp., quasi-)tree and $\mathsf{Var}(\Gamma) \cup \{x\} \subseteq \mathsf{Var}(\mathbf{R})$.*

▶ **Lemma 5.3** (Lifting lemma). *Suppose $S$ is quasi-tree-like. Let $\mathcal{I}$ be interpretation of $S$ into $\mathcal{B}$, $x \in \mathsf{Var}(S)$ and $w \geq \mathcal{I}(x)$. There is an interpretation $\mathcal{I}'$ of $S$ into $\mathcal{B}$ such that $\mathcal{I}'(x) = w$ and for all $y \in \mathsf{Var}(S)$ we have $\mathcal{I}'(y) \geq \mathcal{I}(y)$.*

Let us employ some conventions on $\infty$-proofs. Note that for every inference rule of $\ell\mathsf{IK4}$, except for $\mathsf{th}$ and $\mathsf{cut}$, the premiss(es) are quasi-tree-like whenever the conclusion is. By Observation 3.5 we shall duly assume that $\mathsf{th}$ is not used. Note that this forces relational contexts to be *growing*, bottom-up: the relational context of a sequent always contains those below it. If a cut-formula $z : C$ has label $z$ that does not occur in the conclusion, we may safely rename $z$ to a variable that does. Thus we may assume that any $\infty$-proof in $\ell\mathsf{IGL}$ of $x : A$ has only quasi-tree-like sequents in it.

▶ **Proposition 5.4** (Local soundness). *Suppose $S$ is quasi-tree-like. Let $\mathcal{I}$ be an interpretation of $S$ into $\mathcal{B}$ such that $\mathcal{B}, \mathcal{I} \nvDash S$. For any inference step of $\ell\mathsf{IK4} \setminus \{\mathsf{th}\}$ that $S$ concludes, there is a premiss $S'$ and an interpretation $\mathcal{I}'$ of $S'$ into $\mathcal{B}$ such that $\mathcal{B}, \mathcal{I}' \nvDash S'$ and $\mathcal{I}'(z) \geq \mathcal{I}(z)$ for all $z \in \mathsf{Var}(S)$.*

The proof uses some direct calculations in most cases and the lifting lemma (Lemma 5.3) to handle rules $\to$-$r$ and $\Box$-$r$. We refer to the associated preprint [13] for the case analysis. From here, as in the classical setting for $\ell\mathsf{GL}$, we can employ a contradiction argument by infinite descent to conclude:

▶ **Theorem 5.5** (Soundness). *Suppose $S$ is quasi-tree-like. Then $\ell\mathsf{IGL} \vdash S$ implies $\mathscr{B}\mathsf{IGL} \vDash S$.*

▶ **Corollary 5.6.** *If $\ell\mathsf{IGL} \vdash x : A$ then $\mathscr{B}\mathsf{IGL} \vDash A$.*

The remainder of this work is devoted to proving the converse result. The sections that follow structure the proof into the three parts according to the arrows indicated in Figure 1.

## 6 From birelational models to Kripke predicate models

Towards our countermodel construction in the next section we turn to *predicate models*, essentially via the standard translation, whose internal structure is richer than that of birelational models, thus providing useful invariants for the sequel.

▶ **Definition 6.1** (Predicate models). *A **Kripke structure** is a tuple*

$$(W, \leq, \{D_w\}_{w \in W}, \{\mathsf{Pr}_w\}_{w \in W}, \{R_w\}_{w \in W}), \text{ where}$$

- *$W$ is a non-empty set of* worlds *partially ordered by $\leq$;*
- *$\{D_w\}_{w \in W}$ is a family of non-empty **domains**, such that $D_w \subseteq D_{w'}$ whenever $w \leq w'$;*
- *$\{\mathsf{Pr}_w\}_{w \in W}$ is a family of mappings $\mathsf{Pr}_w : \mathsf{Pr} \to \mathcal{P}(D_w)$ such that for each $p \in \mathsf{Pr}$, $\mathsf{Pr}_w(p) \subseteq \mathsf{Pr}_{w'}(p)$ whenever $w \leq w'$;*
- *$\{R_w\}_{w \in W}$ is a family of relations $R_w \subseteq D_w \times D_w$ such that $R_w \subseteq R_{w'}$ whenever $w \leq w'$.*

▶ **Definition 6.2** (Environment). *Let $\mathcal{K}$ be a Kripke structure. A $w$-**environment** is a function $\rho : \mathsf{Var} \to D_w$.*

Note that a $w$-environment is also a $w'$-environment for any $w' \geq w$.

▶ **Definition 6.3** (Satisfaction). *For modal formula $A$, structure $\mathcal{K}$ and $w$-environment $\rho$, we inductively define the judgement $\mathcal{K}, w \vDash^\rho x : A$ as follows, where $\rho[x := d]$ is the map that sends variable $x$ to $d$ and agrees with $\rho$ on all other variables:*

- $\mathcal{K}, w \vDash^\rho x : p$ *if* $\rho(x) \in \mathsf{Pr}_w(p)$;
- $\mathcal{K}, w \nvDash^\rho \bot$;
- $\mathcal{K}, w \vDash^\rho x : A \wedge B$ *if* $\mathcal{K}, w \vDash^\rho x : A$ *and* $\mathcal{K}, w \vDash^\rho x : B$;
- $\mathcal{K}, w \vDash^\rho x : A \vee B$ *if* $\mathcal{K}, w \vDash^\rho x : A$ *or* $\mathcal{K}, w \vDash^\rho x : B$;
- $\mathcal{K}, w \vDash^\rho x : A \to B$ *if for all* $w' \geq w$, *if* $\mathcal{K}, w' \vDash^\rho x : A$ *then* $\mathcal{K}, w' \vDash^\rho x : B$;
- $\mathcal{K}, w \vDash^\rho x : \Box A$ *if for all* $w' \geq w$ *and* $d \in D_{w'}$, *if* $\rho(x) R_{w'} d$, *then* $\mathcal{K}, w' \vDash^{\rho[y:=d]} y : A$;
- $\mathcal{K}, w \vDash^\rho x : \Diamond A$ *if there exists* $d \in D_w$ *such that* $\rho(x) R_w d$ *and* $\mathcal{K}, w \vDash^{\rho[y:=d]} y : A$.

*We write* $\mathcal{K} \vDash x : A$ *if* $\mathcal{K}, w \vDash^\rho x : A$ *for all worlds* $w$ *and* $w$*-environments* $\rho$.

The monotonicity lemma also holds in Kripke structures.

▶ **Lemma 6.4** (Monotonicity lemma). *Let $\mathcal{K}$ be a Kripke structure. If $w \leq w'$ and $\mathcal{K}, w \vDash^\rho x : A$, then $\mathcal{K}, w' \vDash^\rho x : A$.*

To capture transitivity in Kripke structures, it is sufficient to require each $R_w$ to be transitive, which can be considered as a local condition on Kripke structures. However interpreting termination requires us to consider the interactions between $\leq$ and $R_w$ similarly to the previous section.

Let $\mathcal{K}$ be a Kripke structure. We write $D_W$ for the set of ordered pairs of the form $(w, d)$ with $w \in W$ and $d \in D_w$. We define the two binary relations $\leq_{D_W}, R_{D_W} \subseteq D_W \times D_W$ as:

- $(w, d) \leq_{D_W} (w', d')$ iff $w \leq w'$ and $d = d'$;
- $(w, d) R_{D_W} (w', d')$ iff $w = w'$ and $d R_w d'$.

▶ **Definition 6.5.** *Write $\mathscr{P}\mathsf{IGL}$ for the class of Kripke structures $\mathcal{K}$ satisfying the following:*

- *for all* $w \in W$, $R_w$ *is* transitive; *and*
- *relation* $(\leq_{D_W}; R_{D_W})$ *is* terminating, *i.e., there are no infinite paths* $(w_1, d_1) \leq_{D_W} (w_2, d_1) R_{D_W} (w_2, d_2) \leq_{D_W} (w_3, d_2) R_{D_W} (w_3, d_3) \ldots$.

*We write $\mathscr{P}\mathsf{IGL} \vDash A$ to mean that $\mathcal{K} \vDash x : A$ for all $\mathcal{K} \in \mathscr{P}\mathsf{IGL}$ and any $x \in \mathsf{Var}$.*

The same construction converting a predicate structure into a birelational model from [34, Section 8.1.1] can be used to prove the following result.

▶ **Proposition 6.6.** *If $\mathscr{B}\mathsf{IGL} \vDash A$ then $\mathscr{P}\mathsf{IGL} \vDash A$.*

## 7 Completeness of a multi-succedent calculus via determinacy

Towards completeness we perform a countermodel construction using a proof search strategy based on an intuitionistic *multi-succedent* calculus. This is inspired by analogous arguments for intuitionistic predicate logic, e.g. in [37], but adapted to a non-wellfounded setting.

▶ **Definition 7.1** (Multi-succedent intuitionistic calculus). *The system $\mathsf{m}\ell\mathsf{IK4}$ is the restriction of $\ell\mathsf{K4}$ where the $\Box$-right and $\to$-right rules must have exactly one formula on the RHS. We write $\mathsf{m}\ell\mathsf{IGL}$ for the class of $\infty$-proofs of $\mathsf{m}\ell\mathsf{IK4}$.*

The remainder of this section is devoted to proving the following completeness result:

▶ **Theorem 7.2.** *If $\mathscr{P}\mathsf{IGL} \vDash A$ then $\mathsf{m}\ell\mathsf{IGL} \vdash x : A$.*

Here we informally describe the construction of the proof search tree. For a more formal treatment of parts below we refer to the associated preprint [13]. During bottom-up proof search we will always proceed according to the three following phases in order of priority: applications of rule $\mathsf{tr}$, applications of invertible rules (other than $\mathsf{tr}$), and application of non-invertible rules. The first two together we call the **invertible phase**, the other the **non-invertible phase**. In fact, invertibility of all rules except $\Box$-$r$ and $\to$-$r$ is guaranteed by applying suitable contractions at the same time. For instance, we apply the following derivable 'macro' rules for $\to$ on the left and $\Diamond$ on the right:

$$\to\text{-}l\ \frac{\mathbf{R}, \Gamma, x : A \to B \Rightarrow \Delta, x : A \quad \mathbf{R}, \Gamma, x : A \to B, x : B \Rightarrow \Delta}{\mathbf{R}, \Gamma, x : A \to B \Rightarrow \Delta} \qquad \Diamond\text{-}r\ \frac{\mathbf{R}, xRy, \Gamma \Rightarrow \Delta, x : \Diamond A, y : A}{\mathbf{R}, xRy, \Gamma \Rightarrow \Delta, x : \Diamond A}$$

By furthermore building weakening into the identity, i.e. allowing initial sequents of form $\mathbf{R}, \Gamma, x : p \Rightarrow \Delta, x : p$, structural rules become redundant for proof search. In the invertible phase we can apply the rules in any order. The non-invertible phase creates predecessor nodes for each possible rule instance of $\to$-$r$ and $\Box$-$r$.

In order to carry out our countermodel construction to show completeness of $\mathsf{m\ell IGL}$, we rely on two features of the proof search space that digress from usual countermodel constructions in intuitionistic predicate logic [37]. The first important feature of this strategy is that the invertible phase is always finite and ends in so-called *saturated sequents*, i.e., sequents for which any bottom-up rule application (other than $\mathsf{id}$ and $\bot$-$l$) yields a premiss that is the same sequent, up to multiplicities (see [13] for formal definition). Looking ahead to the countermodel, worlds $w$ are defined on the basis of invertible phases and will as a result all have a finite domain $D_w$.

▶ **Lemma 7.3.** *Following the proof search strategy described above, each invertible phase constructs a finite subtree that has saturated sequents at its leaves.*

Secondly, and perhaps more importantly, we employ a technique from non-wellfounded proof theory to help us organise the countermodel constructed from a failed proof search: we appeal to determinacy of a *proof search game*. To understand the motivation here, a classical countermodel-from-failed-proof search argument proceeds (very roughly) as follows: (1) assume a formula is not provable; (2) for each rule instance there must be an unprovable premiss; (3) continue in this way to construct an (infinite) 'unprovable' branch; (4) extract a countermodel from this branch. In our setting we will need the branch obtained through the process above to be *not progressing* in order to deduce that the structure we extract is indeed one of $\mathscr{P}\mathsf{IGL}$. However the local nature of the process above does not at all guarantee that this will be the case. For this, we rely on *(lightface) analytic determinacy*, which is equivalent to the existence of $0^\sharp$ over ZFC [19], of the corresponding proof search game. As we only use it as a tool and it is not the main focus of our work, we refer to the associated preprint [13] for more details.

▶ **Proposition 7.4.** *Given an unprovable sequent $S$ there is a subtree $T$ of the proof search space rooted at $S$, closed under bottom-up non-invertible rule application*[4] *such that each infinite branch of $T$ is not progressing.*

The properties in Lemma 7.3 and Proposition 7.4 enable us to construct a countermodel:

▶ **Theorem 7.5** (Countermodel construction)**.** *If $\mathsf{m\ell IGL} \nvdash \mathbf{R}, \Gamma \Rightarrow \Delta$, then there is a structure $\mathcal{K} \in \mathscr{P}\mathsf{IGL}$ with $w$-environment $\rho$ such that for labelled formulas $x : A$ we have*

[4] I.e. if $S_0 \in T$ concludes some non-invertible step with premiss $S_1$, then also $S_1 \in T$.

- *if* $x : A \in \Gamma$, *then* $\mathcal{K}, w \models^\rho x : A$, *and,*
- *if* $x : A \in \Delta$, *then* $\mathcal{K}, w \not\models^\rho x : A$.

From here Theorem 7.2 easily follows. Notice that we did not use rule cut in the proof search strategy so we can actually conclude a stronger result:

▶ **Corollary 7.6.** m$\ell$IGL *is* cut-free *complete over* $\mathscr{P}$IGL.

## 8 Completeness of $\ell$IGL via (partial) cut-elimination

To obtain completeness of $\ell$IGL for $\mathscr{B}$IGL, we will simulate m$\ell$IGL using cuts in an extension of $\ell$IGL that allows reasoning over disjunctions of labelled formulas. We apply a (partial) cut-elimination procedure to eliminate these disjunctions, and then note that any resulting proof is already one of $\ell$IGL.

We shall use metavariables $\varphi, \psi$ etc. to vary over *disjunctions* of labelled formulas. I.e. $\varphi, \psi, \ldots \quad ::= \quad (x : A) \quad | \quad \varphi \vee \psi$.

▶ **Definition 8.1.** *The system* $\vee\ell$IK4 *is the extension of* $\ell$IK4 *by duly adapting identity, cut, structural and* $\vee$ *rules to allow for* $\varphi$*-formulas. In particular it has the following* $\vee$ *rules:*

$$\vee\text{-}l\ \frac{\mathbf{R}, \Gamma, \varphi_0 \Rightarrow \psi \qquad \mathbf{R}, \Gamma, \varphi_1 \Rightarrow \psi}{\mathbf{R}, \Gamma, \varphi_0 \vee \varphi_1 \Rightarrow \psi} \qquad \vee\text{-}r\ \frac{\mathbf{R}, \Gamma \Rightarrow \varphi_i}{\mathbf{R}, \Gamma \Rightarrow \varphi_0 \vee \varphi_1}\ i \in \{0, 1\}$$

The **degree** of a formula $(x_1 : A_1) \vee \cdots \vee (x_d : A_d)$ is $d$. The degree of a cut is the degree $d$ of its cut-formula, in which case we say it is a $d$-cut. The degree of a preproof is the maximum degree of its cuts (when this is well-defined).

▶ **Proposition 8.2.** *If* m$\ell$IK4 *has a cut-free* $\infty$*-proof of* $x : A$*,* $\vee\ell$IK4 *has one of bounded degree.*

Moreover, immediately from definitions, we have:

▶ **Observation 8.3.** *A* $\vee\ell$IK4 $\infty$*-proof containing only labelled formulas is a* $\ell$IK4 $\infty$*-proof.*

Thus, to conclude completeness of $\ell$IGL for $\mathscr{B}$IGL, it suffices to eliminate the use of disjunctions of labelled formulas in $\vee\ell$IK4 $\infty$-proofs. We will prove this by a partial cut-elimination procedure, reducing cuts over disjunctions of labelled formulas until they are on labelled formulas.

For the remainder of this section we work only with preproofs without thinning th, by Observation 3.5. Recall that this means that relational contexts are *growing*, bottom-up. For a sequent $S$ write $\mathbf{R}_S$ for its relational context. I.e. if $S$ is $\mathbf{R}, \Gamma \Rightarrow \varphi$, then $\mathbf{R}_S := \mathbf{R}$.

▶ **Lemma 8.4** (Invertibility). *If* $\vee\ell$IK4 *has a* $\infty$*-proof* $P$ *of* $\mathbf{R}, \Gamma, \varphi_0 \vee \varphi_1 \Rightarrow \psi$ *then it also has* $\infty$*-proofs* $P_i$ *of* $\mathbf{R}, \Gamma, \varphi_i \Rightarrow \psi$*, for* $i \in \{0, 1\}$*. Moreover, for each branch* $(S_i)_{i<\omega}$ *of* $P_i$ *there is a branch* $(S_i')_{i<\omega}$ *of* $P$ *such that* $\mathbf{R}_{S_i} \subseteq \mathbf{R}_{S_i'}$ *for all* $i < \omega$*.*

From here the key cut-reduction for $\varphi$-formulas is simply,

$$\text{cut}\ \frac{\vee\text{-}r\ \dfrac{\overset{P}{\vdots}}{\dfrac{\mathbf{R}, \Gamma, \Rightarrow \chi_i}{\mathbf{R}, \Gamma \Rightarrow \chi_0 \vee \chi_1}} \qquad \overset{Q}{\vdots}\ \ \mathbf{R}, \Gamma', \chi_0 \vee \chi_1 \Rightarrow \varphi}{\mathbf{R}, \Gamma, \Gamma' \Rightarrow \varphi} \quad \rightsquigarrow \quad \text{cut}\ \frac{\overset{P}{\vdots}\ \ \mathbf{R}, \Gamma \Rightarrow \chi_i \qquad \overset{Q_i}{\vdots}\ \ \mathbf{R}, \Gamma', \chi_i \Rightarrow \varphi}{\mathbf{R}, \Gamma, \Gamma' \Rightarrow \varphi}$$

where $Q_i$ is obtained by Lemma 8.4 above. Note that this reduction 'produces' a cut of lower complexity. Commutative cut-reduction cases, where the cut-formula is not principal on the left, are standard and always produce. Note that, thanks to invertibility, we do not consider commutations over the right premiss of a $\varphi$-cut. However commutative cases may *increase* the heights of progress points; for instance when commuting over a $\Box$-$l$ step,

$$
\text{cut}\ \dfrac{\Box\text{-}l\ \dfrac{\overset{P}{\mathbf{R}, xRy, \Gamma, y : A \Rightarrow \chi}}{\mathbf{R}, xRy, \Gamma, x : \Box A \Rightarrow \chi} \quad \overset{Q}{\mathbf{R}, xRy, \Gamma', \chi \Rightarrow \varphi}}{\mathbf{R}, xRy, \Gamma, \Gamma', x : \Box A \Rightarrow \varphi}
\quad \rightsquigarrow \quad
\Box\text{-}l\ \dfrac{\text{cut}\ \dfrac{\overset{P}{\mathbf{R}, xRy, \Gamma, y : A \Rightarrow \chi} \quad \overset{Q}{\mathbf{R}, xRy, \Gamma', \chi \Rightarrow \varphi}}{\mathbf{R}, xRy, \Gamma, \Gamma', y : A \Rightarrow \varphi}}{\mathbf{R}, xRy, \Gamma, \Gamma', x : \Box A \Rightarrow \varphi}
$$

observe that progress points in $Q$ have been raised. Thus, to show that the limit of cut-reduction is progressing we will need appropriate invariants, requiring additional notions.

A **bar** of a preproof is a (necessarily finite, by König's Lemma) antichain intersecting each infinite branch. Each bar $B$ induces a (necessarily finite) subtree $\lfloor B \rfloor$ of nodes beneath (and including) it. Given a cut-reduction $P \rightsquigarrow_{\mathsf{r}} P'$ we associate to each bar $B$ of $P$ a bar $\mathsf{r}(B)$ of $P'$ in the natural way. In particular we have that $\mathsf{r}(B)$ satisfies the following properties:

▶ **Lemma 8.5** (Trace preservation). *If $P \rightsquigarrow_{\mathsf{r}} P'$ and $B$ a bar of $P$, for any sequent $S' \in \mathsf{r}(B)$ there is a sequent $S \in B$ such that $\mathbf{R}_{S'} \supseteq \mathbf{R}_S$.*

This lemma allows us to keep track of a fixed amount of progress information during cut-elimination, sidestepping the issue that commutative cases raise progress points. Note that we really need the $\mathbf{R}_{S'} \supseteq \mathbf{R}_S$ due to the $\Diamond$-$l$ commutative case:

$$
\text{cut}\ \dfrac{\Diamond\text{-}l\ \dfrac{\overset{P}{\mathbf{R}, xRy, \Gamma, y : A \Rightarrow \chi}}{\mathbf{R}, \Gamma, x : \Diamond A \Rightarrow \chi} \quad \overset{Q}{\mathbf{R}, \Gamma', \chi \Rightarrow \varphi}}{\mathbf{R}, \Gamma, \Gamma', x : \Diamond A \Rightarrow \varphi}
\quad \rightsquigarrow \quad
\Diamond\text{-}l\ \dfrac{\text{cut}\ \dfrac{\overset{P}{\mathbf{R}, xRy, \Gamma, y : A \Rightarrow \chi} \quad \overset{xRy, Q}{xRy, \mathbf{R}, \Gamma', \chi \Rightarrow \varphi}}{\mathbf{R}, xRy, \Gamma, \Gamma', y : A \Rightarrow \varphi}}{\mathbf{R}, \Gamma, \Gamma', x : \Diamond A \Rightarrow \varphi}
$$

where the preproof $xRy, Q$ is obtained from $Q$ by prepending $xRy$ to the LHS of each sequent. From here we can effectively reduce infinitary cut-elimination to finitary cut-elimination, by eliminating cuts beneath higher and higher bars. The step case is given by:

▶ **Lemma 8.6** (Productivity). *For any $\infty$-proof $P$ and bar $B$ there is a sequence of cut-reductions $P = P_0 \rightsquigarrow_{\mathsf{r}_1} \cdots \rightsquigarrow_{\mathsf{r}_n} P_n$ such that $\mathsf{r}_n \cdots \mathsf{r}_1(B)$ has no $d$-cuts beneath it in $P_n$.*

The argument proceeds in a relatively standard way: by induction on the number of $d$-cuts in $\lfloor B \rfloor$, with a subinduction on the multiset of the distances of topmost $d$-cuts in $\lfloor B \rfloor$ from $B$, where the distance is the length of the shortest path from the cut to $B$. Applying Lemma 8.6 to higher and higher bars allows us to reduce cut-degrees as required:

▶ **Proposition 8.7** (Degree-reduction). *For each $\infty$-proof of $\vee\ell\mathsf{IK4}$ of $x : A$ of degree $d$, there is one of degree $< d$.*

Importantly here, the preservation of progress in the limit crucially relies on Lemma 8.5, under König's Lemma. Finally by induction on degree we have:

▶ **Corollary 8.8** (Partial cut-elimination). *If $\vee\ell\mathsf{IK4}$ has a bounded-degree $\infty$-proof of $x : A$, then it has one containing only labelled formulas.*

From here we have our desired converse to Theorem 5.5, following from Observation 8.3 and Propositions 6.6 and 8.2, Theorem 7.2 and Corollary 8.8:

▶ **Theorem 8.9** (Completeness). *If* $\mathscr{B}\mathsf{IGL} \models A$ *then* $\ell\mathsf{IGL} \vdash x : A$*.*

## 9 Conclusions

We have recovered several intuitionistic formulations of GL, both syntactically and semantically, in the tradition of Simpson, on intuitionistic modal logic [34], and Simpson, on non-wellfounded proofs [35]. We proved the equivalence of all these formulations, cf. Figure 1, motivating the following definition:

▶ **Definition 9.1.** IGL *is the modal logic given by any/all of the nodes of Figure 1.*

Thanks to the methodology we followed, IGL satisfies Simpson's *requirements* from [34]. IGL also interprets (classical) GL along the Gödel-Gentzen negative translation.

It would be interesting to examine IGL as a logic of provability, returning to the origins of GL. In particular IGL (even IK) has a *normal* $\Diamond$, distributing over $\vee$, so let us point out that it is not sound to interpret $\Diamond$ as consistency $\neg\Box\neg$. At the same time (effective) model-theoretic readings of the $\Diamond$ stumble on the consequence (already of IK) $\Box(A \to B) \to \Diamond A \to \Diamond B$. We expect IGL to rather correspond to the provability logic of some *model* of HA.

To this end it would be pertinent to develop a bona fide *axiomatisation* of IGL. As far as we know it is possible that IGL is simply the extension of IK4 by Löb's axiom but, similarly to other 'I' logics, we expect that further axioms involving $\Diamond$ will be necessary. On a related note, we believe that a full cut-elimination result holds for $\ell$IGL, in particular by extending our argument for $\vee\ell$IGL. Crucial here is that no cut-reductions *delete* progress points, since we do not cut on relational atoms (cf. a key $\Box$ reduction). A full development of this is beyond the scope (and allocated space) of this work.

## References


1 Mohammad Ardeshir and Motjaba Mojtahedi. The $\Sigma_1$-provability logic of HA. *Annals of Pure and Applied Logic*, 169(10):997–1043, 2018.

2 David Baelde, Amina Doumane, and Alexis Saurin. Infinitary proof theory: the multiplicative additive case. In *25th EACSL Annual Conference on Computer Science Logic, CSL 2016*, volume 62 of *LIPIcs*, pages 42:1–17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016.

3 Stefano Berardi and Makoto Tatsuta. Equivalence of inductive definitions and cyclic proofs under arithmetic. In *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–12. IEEE, 2017.

4 Lars Birkedal, Rasmus Ejlers Møgelberg, Jan Schwinghammer, and Kristian Støvring. First steps in synthetic guarded domain theory: step-indexing in the topos of trees. *Logical Methods in Computer Science*, 8, 2012.

5 Patrick Blackburn, Maarten De Rijke, and Yde Venema. *Modal logic*, volume 53. Cambridge University Press, 2001.

6 Patrick Blackburn, Johan van Benthem, and Frank Wolter, editors. *Handbook of Modal Logic*, volume 3 of *Studies in logic and practical reasoning*. North-Holland, 2007.

7 James Brotherston and Alex Simpson. Sequent calculi for induction and infinite descent. *Journal of Logic and Computation*, 21(6):1177–1216, 2011.

8 Ranald Clouston and Rajeev Goré. Sequent calculus in the topos of trees. In *Foundations of Software Science and Computation Structures: 18th International Conference, FOSSACS 2015*, pages 133–147. Springer, 2015.

9 Anupam Das. On the logical complexity of cyclic arithmetic. *Logical Methods in Computer Science*, 16, 2020.

10 Anupam Das and Sonia Marin. Brouwer meets Kripke: constructivising modal logic, 2022. Post on *The Proof Theory Blog* (accessed 2 August 2023). `https://prooftheory.blog/2022/08/19/brouwer-meets-kripke-constructivising-modal-logic/`.

11 Anupam Das and Sonia Marin. On intuitionistic diamonds (and lack thereof). In *Automated Reasoning with Analytic Tableaux and Related Methods*, volume 14278 of *Lecture Notes in Computer Science*, pages 283–301. Springer, 2023.

12 Anupam Das and Damien Pous. Non-wellfounded proof theory for (kleene+action)(algebras+lattices). In *27th EACSL Annual Conference on Computer Science Logic, CSL 2018*, volume 119 of *LIPIcs*, pages 19:1–19:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018.

13 Anupam Das, Iris van der Giessen, and Sonia Marin. Intuitionistic gödel-löb logic, à la simpson: labelled systems and birelational semantics, 2023. `arXiv:2309.00532`.

14 Gisèle Fischer Servi. On modal logic with an intuitionistic base. *Studia Logica*, 36:141–149, 1977.

15 Iris van der Giessen. *Uniform Interpolation and Admissible Rules: Proof-theoretic investigations into (intuitionistic) modal logics*. PhD thesis, Utrecht University, 2022.

16 Iris van der Giessen and Rosalie Iemhoff. Sequent calculi for intuitionistic Gödel–Löb logic. *Notre Dame Journal of Formal Logic*, 62(2):221–246, 2021.

17 Rajeev Goré and Revantha Ramanayake. Valentini's cut-elimination for provability logic resolved. In Carlos Areces and Robert Goldblatt, editors, *Proceedings of the 7th conference on Advances in Modal Logic*, pages 67–86. College Publications, 2008.

18 Rajeev Goré and Ian Shillito. Direct elimination of additive-cuts in GL4ip: verified and extracted. In *Advances in Modal Logic 14*, 2022.

19 Leo Harrington. Analytic determinacy and $0^{\#}$. *The Journal of Symbolic Logic*, 43(4):685–693, 1978. `doi:10.2307/2273508`.

20 Rosalie Iemhoff. Reasoning in circles. In Jan van Eijck, Joost J. Joosten, and Rosalie Iemhoff, editors, *Liber Amicorum Alberti. A Tribute to Albert Visser*, pages 165–178. College Publications, 2016.

21 G. Alex Kavvos. Intensionality, intensional recursion and the Gödel-Löb axiom. *FLAP*, 8(8):2287–2312, 2021.

22 Daniel Leivant. On the proof theory of the modal logic for arithmetic provability. *The Journal of Symbolic Logic*, 46(3):531–538, 1981.

23 Tadeusz Litak. Constructive modalities with provability smack. In Guram Bezhanishvili, editor, *Leo Esakia on Duality in Modal and Intuitionistic Logics*, volume 4 of *Outstanding Contributions to Logic*, pages 187–216. Springer Netherlands, 2014.

24 Anders Moen. The proposed algorithms for eliminating cuts in the provability calculus GLS do not terminate. In *The 13th Nordic Workshop in Programming Theory*, 2001.

25 Mojtaba Mojtahedi. On provability logic of HA, 2022. URL: `https://doi.org/10.48550/arXiv.2206.00445`, `arXiv:2206.00445`.

26 Hiroshi Nakano. A modality for recursion. In *Proceedings Fifteenth Annual IEEE Symposium on Logic in Computer Science (Cat. No. 99CB36332)*, pages 255–266. IEEE, 2000.

27 Sara Negri. Proof analysis in modal logic. *Journal of Philosophical Logic*, 34:507–544, 2005.

28 Damian Niwiński and Igor Walukiewicz. Games for the $\mu$-calculus. *Theoretical Computer Science*, 163(1-2):99–116, 1996.

29 Gordon Plotkin and Colin Stirling. A framework for intuitionistic modal logics. In *Proceedings of the 1st Conference on Theoretical Aspects of Reasoning about Knowledge (TARK)*, pages 399–406, 1986.

30 Francesca Poggiolesi. A purely syntactic and cut-free sequent calculus for the modal logic of provability. *The Review of Symbolic Logic*, 2(4):593–611, 2009.

31 Katsumi Sasaki. Löb's axiom and cut-elimination theorem. *Academia Mathematical Sciences and Information Engineering Nanzan University*, 1:91–98, 2001.

32 Krister Segerberg. Results in non-classical propositional logic. *Uppsala: Filosofiska Föreningen och Filosofiska Institutionen vid Uppsala Universitet.*, 1971.

33 Daniyar S. Shamkanov. Circular proofs for the Gödel-Löb provability logic. *Mathematical Notes*, 96:575–585, 2014.

34 Alex K. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logic*. PhD thesis, University of Edinburgh, 1994.

35 Alex K. Simpson. Cyclic arithmetic is equivalent to Peano arithmetic. In *Foundations of Software Science and Computation Structures Proceedings*, volume 10203 of *Lecture Notes in Computer Science*, pages 283–300, 2017.

36 Robert M. Solovay. Provability interpretations of modal logic. *Israel Journal of Mathematics*, 25:287–304, 1976.

37 Gaisi Takeuti. *Proof Theory*. New York, N.Y., U.S.A.: Sole distributors for the U.S.A. and Canada, Elsevier Science Pub. Co., 1975.

38 Silvio Valentini. The modal logic of provability: cut-elimination. *Journal of Philosophical logic*, pages 471–476, 1983.