Ecumenical modal logic*

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Abstract. The discussion about how to put together Gentzen's systems for classical and intuitionistic logic in a single unified system is back in fashion. Indeed, recently Prawitz and others have been discussing the so called ecumenical Systems, where connectives from these logics can co-exist in peace. In Prawitz' system, the classical logician and the intuitionistic logician would share the universal quantifier, conjunction, negation, and the constant for the absurd, but they would each have their own existential quantifier, disjunction, and implication, with different meanings. Prawitz' main idea is that these different meanings are given by a semantical framework that can be accepted by both parties. In this work we extend Prawitz' ecumenical idea to alethic K-modalities.

1 Introduction

In [17] Dag Prawitz proposed a natural deduction system for what was later called *ecumenical logic* (EL), where classical and intuitionistic logic could coexist in peace. In this system, the classical logician and the intuitionistic logician would share the universal quantifier, conjunction, negation, and the constant for the absurd (*the neutral connectives*), but they would each have their own existential quantifier, disjunction, and implication, with different meanings. Prawitz' main idea is that these different meanings are given by a semantical framework that can be accepted by both parties. While proof-theoretical aspects were also considered, his work was more focused on investigating the philosophical significance of the fact that classical logic can be translated into intuitionistic logic.

Pursuing the idea of having a better understanding of ecumenical systems under the proof-theoretical point of view, in [15] an ecumenical sequent calculus (LEci) was proposed. This enabled not only the proof of some important proof theoretical properties (such as cut-elimination and invertibility of rules), but it also provided a better understanding of the ecumenical nature of consequence: it is intrinsically intuitionistic, being classical only in the presence of classical succedents.

Ecumenism in logic is interesting for different reasons, and we discuss some in Section 7. But maybe the most compelling argument for considering ecumenical

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systems is the analysis of mathematical theories and proofs. In fact, the possibility of having classical and intuitionistic reasoning in a single framework allows determining, for example, which parts of a proof can be done constructively, or which axioms in a theory can be restricted to its intuitionistic formulation. As a simple example, consider the following statement, where $x, y \in \mathbb{R}$:

$$\text{if } x + y = 16 \text{ then } x \geq 8 \text{ or } y \geq 8.$$

Of course, this could always be translated into a classical formula, but a finer analysis shows that the disjunction should definitely be classical, while the implication does not need to. That is, the statement should be translated as $(x + y = 16) \rightarrow_i x \geq 8 \vee_c y \geq 8$.

In this work, we propose lifting this discussion to modal logics, by presenting an extension of EL with the alethic modalities of *necessity* and *possibility*. There are many choices to be made and many relevant questions to be asked, *e.g.*: what is the ecumenical interpretation of ecumenical modalities? Should we add classical, intuitionistic, or neutral versions for modal connectives? What is really behind the difference between the classical and intuitionistic notions of truth?

We propose an answer for these questions in the light of Simpson's meta-logical interpretation of modalities [20] by embedding the expected semantical behavior of the modal operator into the ecumenical first order logic.

We start by highlighting the main proof theoretical aspects of LEci (Section 2). This is vital for understanding how the embedding mentioned above will mold the behavior of ecumenical modalities, since modal connectives are interpreted in first order logics using quantifiers. In Sections 3 and 4, we justify our choices by following closely Simpson's script, with the difference that we prove meta-logical soundness and completeness using proof theoretical methods only. We then provide an axiomatic and semantical interpretation of ecumenical modalities in Section 5. This makes it possible to extend the discussion, in Section 6, to relational systems with the usual restrictions on the relation in the Kripke model. We end the paper with a discussion about logical ecumenism in general.

2 The system LEci

The language $\mathcal{L}$ used for ecumenical systems is described as follows. We will use a subscript c for the classical meaning and i for the intuitionistic, dropping such subscripts when formulae/connectives can have either meaning.

Classical and intuitionistic n-ary predicate symbols $(p_c, p_i, \ldots)$ co-exist in $\mathcal{L}$ but have different meanings. The neutral logical connectives $\{\bot, \neg, \wedge, \forall\}$ are common for classical and intuitionistic fragments, while $\{\rightarrow_i, \vee_i, \exists_i\}$ and $\{\rightarrow_c, \vee_c, \exists_c\}$ are restricted to intuitionistic and classical interpretations, respectively.

The sequent system LEci (depicted in Fig. 1) was presented in [15] as the sequent counterpart of Prawitz' natural deduction system. Observe that the rules R_c and L_c describe the intended meaning of the predicate p_c, from the intuitionistic predicate p_i.

LEci has very interesting proof theoretical properties, together with a Kripke semantical interpretation, that allowed the proposal of a variety of ecumenical

Initial and structural rules

$$\frac{}{A, \Gamma \Rightarrow A}\ \mathsf{init} \qquad \frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow A}\ \mathsf{W}$$

Propositional rules

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ \wedge L \quad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \wedge R \quad \frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee_i B, \Gamma \Rightarrow C}\ \vee_i L \quad \frac{\Gamma \Rightarrow A_j}{\Gamma \Rightarrow A_1 \vee_i A_2}\ \vee_i R_j$$

$$\frac{A, \Gamma \Rightarrow \bot \quad B, \Gamma \Rightarrow \bot}{A \vee_c B, \Gamma \Rightarrow \bot}\ \vee_c L \quad \frac{\Gamma, \neg A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \vee_c B}\ \vee_c R \quad \frac{A \rightarrow_i B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{\Gamma, A \rightarrow_i B \Rightarrow C}\ \rightarrow_i L$$

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow_i B}\ \rightarrow_i R \quad \frac{A \rightarrow_c B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow \bot}{A \rightarrow_c B, \Gamma \Rightarrow \bot}\ \rightarrow_c L \quad \frac{\Gamma, A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \rightarrow_c B}\ \rightarrow_c R$$

$$\frac{\neg A, \Gamma \Rightarrow A}{\neg A, \Gamma \Rightarrow \bot}\ \neg L \quad \frac{\Gamma, A \Rightarrow \bot}{\Gamma \Rightarrow \neg A}\ \neg R \quad \frac{}{\bot, \Gamma \Rightarrow A}\ \bot L \quad \frac{p_i, \Gamma \Rightarrow \bot}{p_c, \Gamma \Rightarrow \bot}\ L_c \quad \frac{\Gamma, \neg p_i \Rightarrow \bot}{\Gamma \Rightarrow p_c}\ R_c$$

Quantifiers

$$\frac{A[y/x], \forall x.A, \Gamma \Rightarrow C}{\forall x.A, \Gamma \Rightarrow C}\ \forall L \quad \frac{\Gamma \Rightarrow A[y/x]}{\Gamma \Rightarrow \forall x.A}\ \forall R$$

$$\frac{A[y/x], \Gamma \Rightarrow C}{\exists_i x.A, \Gamma \Rightarrow C}\ \exists_i L \quad \frac{\Gamma \Rightarrow A[y/x]}{\Gamma \Rightarrow \exists_i x.A}\ \exists_i R \quad \frac{A[y/x], \Gamma \Rightarrow \bot}{\exists_c x.A, \Gamma \Rightarrow \bot}\ \exists_c L \quad \frac{\Gamma, \forall x.\neg A \Rightarrow \bot}{\Gamma \Rightarrow \exists_c x.A}\ \exists_c R$$

Fig. 1. Ecumenical sequent system LEci. In rules $\forall R, \exists_i L, \exists_c L$, the eigenvariable y is fresh.

proof systems, such as multi-conclusion and nested sequent systems, as well as several fragments of such systems [15].

Denoting by $\vdash_{\mathsf{S}} A$ the fact that the formula A is a theorem in the proof system S, the following theorems are easily provable in LEci:

1. $\vdash_{\mathsf{LEci}} (A \rightarrow_c \bot) \leftrightarrow_i (A \rightarrow_i \bot) \leftrightarrow_i (\neg A)$;
2. $\vdash_{\mathsf{LEci}} (A \vee_c B) \leftrightarrow_i \neg(\neg A \wedge \neg B)$;
3. $\vdash_{\mathsf{LEci}} (A \rightarrow_c B) \leftrightarrow_i \neg(A \wedge \neg B)$;
4. $\vdash_{\mathsf{LEci}} (\exists_c x.A) \leftrightarrow_i \neg(\forall x.\neg A)$.

Note that (1) means that the ecumenical system defined in Fig. 1 does not distinguish between intuitionistic or classical negations, thus they can be called simply $\neg A$. We prefer to keep the negation operator in the language since the calculi presented in this work make heavy use of it.

Theorems (2) to (4) are of interest since they relate the classical and the neutral operators: the classical connectives can be defined using negation, conjunction, and the universal quantifier. On the other hand,

5. $\vdash_{\mathsf{LEci}} (A \rightarrow_i B) \rightarrow_i (A \rightarrow_c B)$ but $\not\vdash_{\mathsf{LEci}} (A \rightarrow_c B) \rightarrow_i (A \rightarrow_i B)$ in general;
6. $\vdash_{\mathsf{LEci}} A \vee_c \neg A$ but $\not\vdash_{\mathsf{LEci}} A \vee_i \neg A$ in general;
7. $\vdash_{\mathsf{LEci}} (\neg\neg A) \rightarrow_c A$ but $\not\vdash_{\mathsf{LEci}} (\neg\neg A) \rightarrow_i A$ in general;
8. $\vdash_{\mathsf{LEci}} (A \wedge (A \rightarrow_i B)) \rightarrow_i B$ but $\not\vdash_{\mathsf{LEci}} (A \wedge (A \rightarrow_c B)) \rightarrow_i B$ in general;
9. $\vdash_{\mathsf{LEci}} \forall x.A \rightarrow_i \neg\exists_c x.\neg A$ but $\not\vdash_{\mathsf{LEci}} \neg\exists_c x.\neg A \rightarrow_i \forall x.A$ in general.

Observe that (4) and (9) reveal the asymmetry between definability of quantifiers: while the classical existential can be defined from the universal quantification, the other way around is not true, in general. This is closely related with the fact that, proving $\forall x.A$ from $\neg\exists_c x.\neg A$ depends on A being a classical formula. We will come back to this in Section 3.

On its turn, the following result states that logical consequence in LEci is intrinsically intuitionistic.

Proposition 1 ([15]). *$\Gamma \vdash B$ is provable in* LEci *iff* $\vdash_{\mathsf{LEci}} \bigwedge \Gamma \rightarrow_i B$.

To preserve the "classical behaviour", *i.e.*, to satisfy all the principles of classical logic *e.g. modus ponens* and the *classical reductio*, it is sufficient that the main operator of the formula be classical (see [14]). Thus, "hybrid" formulas, *i.e.*, formulas that contain classical and intuitionistic operators may have a classical behaviour. Formally,

Definition 1. *A formula B is called externally classical (denoted by B^c) if and only if B is $\bot$, a classical predicate letter, or its root operator is classical (that is: $\rightarrow_c, \vee_c, \exists_c$). A formula C is* classical *if it is built from classical atomic predicates using only the connectives: $\rightarrow_c, \vee_c, \exists_c, \neg, \wedge, \forall$, and the unit $\bot$.*

For externally classical formulas we can now prove the following theorems

10. $\vdash_{\mathsf{LEci}} (A \rightarrow_c B^c) \rightarrow_i (A \rightarrow_i B^c)$.
11. $\vdash_{\mathsf{LEci}} (A \wedge (A \rightarrow_c B^c)) \rightarrow_i B^c$.
12. $\vdash_{\mathsf{LEci}} \neg\neg B^c \rightarrow_i B^c$.
13. $\vdash_{\mathsf{LEci}} \neg\exists_c x.\neg B^c \rightarrow_i \forall x.B^c$.

Moreover, notice that all classical right rules as well as the right rules for the neutral connectives in LEci are invertible. Since invertible rules can be applied eagerly when proving a sequent, this entails that classical formulas can be eagerly decomposed. As a consequence, the ecumenical entailment, when restricted to classical succedents (antecedents having an unrestricted form), is classical.

Theorem 1 ([15]). *Let C be a classical formula and Γ be a multiset of ecumenical formulas. Then*

$$\vdash_{\mathsf{LEci}} \bigwedge \Gamma \rightarrow_c C \textit{ iff } \vdash_{\mathsf{LEci}} \bigwedge \Gamma \rightarrow_i C.$$

This sums up well, proof theoretically, the *ecumenism* of Prawitz' original proposal.

3 Ecumenical modalities

In this section we will propose an ecumenical view for alethic modalities. Since there are a number of choices to be made, we will construct our proposal step-by-step.

3.1 Normal modal logics

The language of *(propositional, normal) modal formulas* consists of a denumerable set $\mathcal{P}$ of propositional symbols and a set of propositional connectives enhanced with the unary *modal operators* $\Box$ and $\Diamond$ concerning necessity and possibility, respectively [3].

The semantics of modal logics is often determined by means of *Kripke models.* Here, we will follow the approach in [20], where a modal logic is characterized by the respective interpretation of the modal model in the meta-theory (called *meta-logical characterization*).

Formally, given a variable x, we recall the standard translation $[\cdot]_x$ from modal formulas into first-order formulas with at most one free variable, x, as follows: if p is atomic, then $[p]_x = p(x)$; $[\bot]_x = \bot$; for any binary connective $\star$, $[A \star B]_x = [A]_x \star [B]_x$; for the modal connectives

$$[\Box A]_x = \forall y(R(x,y) \rightarrow [A]_y) \qquad [\Diamond A]_x = \exists y(R(x,y) \wedge [A]_y)$$

where $R(x,y)$ is a binary predicate.

Opening a parenthesis: such a translation has, as underlying justification, the interpretation of alethic modalities in a Kripke model $\mathcal{M} = (W, R, V)$:

$$\begin{array}{lll} \mathcal{M}, w \models \Box A & \text{iff} & \text{for all } v \text{ such that } wRv, \mathcal{M}, v \models A. \\ \mathcal{M}, w \models \Diamond A & \text{iff} & \text{there exists } v \text{ such that } wRv \text{ and } \mathcal{M}, v \models A. \end{array} \tag{1}$$

$R(x,y)$ then represents the *accessibility relation* R in a Kripke frame. This intuition can be made formal based on the one-to-one correspondence between classical/intuitionistic translations and Kripke modal models [20]. We close this parenthesis by noting that this justification is only motivational, aiming at introducing modalities. Models will be discussed formally in Section 5.1.

The object-modal logic OL is then characterized in the first-order meta-logic ML as

$$\vdash_{OL} A \quad \text{iff} \quad \vdash_{ML} \forall x.[A]_x$$

Hence, if ML is classical logic (CL), the former definition characterizes the classical modal logic K [3], while if it is intuitionistic logic (IL), then it characterizes the intuitionistic modal logic IK [20].

In this work, we will adopt EL as the meta-theory (given by the system LEci), hence characterizing what we will define as the ecumenical modal logic EK.

3.2 An ecumenical view of modalities

The language of *ecumenical modal formulas* consists of a denumerable set $\mathcal{P}$ of (ecumenical) propositional symbols and the set of ecumenical connectives enhanced with unary *ecumenical modal operators*. Unlike for the classical case, there is not a canonical definition of constructive or intuitionistic modal logics. Here we will mostly follow the approach in [20] for justifying our choices for the ecumenical interpretation for *possibility* and *necessity.*

The ecumenical translation $[\cdot]_x^e$ from propositional ecumenical formulas into LEci is defined in the same way as the modal translation $[\cdot]_x$ in the last section. For the case of modal connectives, observe that, due to Proposition 1, the interpretation of ecumenical consequence should be essentially *intuitionistic*. This implies that the box modality is a *neutral connective*. The diamond, on the other hand, has two possible interpretations: classical and intuitionistic, since its leading connective is an existential quantifier. Hence we should have the ecumenical modalities: $\Box, \Diamond_i, \Diamond_c$, determined by the translations

$$[\Box A]_x^e = \forall y(R(x,y) \to_i [A]_y^e)$$
$$[\Diamond_i A]_x^e = \exists_i y(R(x,y) \wedge [A]_y^e) \qquad [\Diamond_c A]_x^e = \exists_c y(R(x,y) \wedge [A]_y^e)$$

Observe that, due to the equivalence (4), we have

14. $\Diamond_c A \leftrightarrow_i \neg\Box\neg A$

On the other hand, $\Box$ and $\Diamond_i$ are not inter-definable due to (9). Finally, if A^c is externally classical, then

15. $\Box A^c \leftrightarrow_i \neg\Diamond_c\neg A^c$

This means that, when restricted to the classical fragment, $\Box$ and $\Diamond_c$ are duals. This reflects well the ecumenical nature of the defined modalities. We will denote by EK the ecumenical modal logic meta-logically characterized by LEci via $[\cdot]_x^e$.

4 A labeled system for EK

The basic idea behind labeled proof systems for modal logic is to internalize elements of the associated Kripke semantics (namely, the worlds of a Kripke structure and the accessibility relation between them) into the syntax.

Labeled modal formulas are either *labeled formulas* of the form $x : A$ or *relational atoms* of the form xRy, where x, y range over a set of variables and A is a modal formula. *Labeled sequents* have the form $\Gamma \vdash x : A$, where Γ is a multiset containing labeled modal formulas. The labeled ecumenical system labEK is presented in Fig. 2.

We will denote by $[\Gamma \vdash x : A]$ the LEci sequent $[\Gamma] \Rightarrow [A]_x^e$ where $[\Gamma] = \{R(x,y) \mid xRy \in \Gamma\} \cup \{[B]_x^e \mid x : B \in \Gamma\}$. The following is a meta-logical soundness and completeness result.

Theorem 2. *The following are equivalent:*

1. $\Gamma \vdash x : A$ *is provable in* labEK.
2. $[\Gamma \vdash x : A]$ *is provable in* LEci.

Proof. We will consider the following translation between labEK rule applications and LEci derivations, where the translation for the propositional rules is the trivial one:

$$\frac{xRy, y : A, x : \Box A, \Gamma \vdash z : C}{xRy, x : \Box A, \Gamma \vdash z : C}\ \Box L \quad \rightsquigarrow$$

$$\dfrac{R(x,y), [\Box A]_x^e, R(x,y) \to_i [A]_y^e, [\Gamma] \Rightarrow R(x,y) \quad [xRy, y : A, x : \Box A, \Gamma \vdash z : C]}{[xRy, x : \Box A, \Gamma \vdash z : C]}\ (\forall L, \to_i L)$$

Initial and structural rules

$$\frac{}{x:A,\Gamma\vdash x:A}\ \mathsf{init}\qquad \frac{\Gamma\vdash y:\bot}{\Gamma\vdash x:A}\ \mathsf{W}$$

Propositional rules

$$\frac{x:A,x:B,\Gamma\vdash z:C}{x:A\wedge B,\Gamma\vdash z:C}\ \wedge L\qquad \frac{\Gamma\vdash x:A\quad \Gamma\vdash x:B}{\Gamma\vdash x:A\wedge B}\ \wedge R$$

$$\frac{x:A,\Gamma\vdash z:C\quad x:B,\Gamma\vdash z:C}{x:A\vee_i B,\Gamma\vdash z:C}\ \vee_i L\qquad \frac{\Gamma\vdash x:A_j}{\Gamma\vdash x:A_1\vee_i A_2}\ \vee_i R_j$$

$$\frac{x:A,\Gamma\vdash x:\bot\quad x:B,\Gamma\vdash x:\bot}{x:A\vee_c B,\Gamma\vdash x:\bot}\ \vee_c L\qquad \frac{\Gamma,x:\neg A,x:\neg B\vdash x:\bot}{\Gamma\vdash x:A\vee_c B}\ \vee_c R$$

$$\frac{x:A\rightarrow_i B,\Gamma\vdash x:A\quad x:B,\Gamma\vdash z:C}{x:A\rightarrow_i B,\Gamma\vdash z:C}\ \rightarrow_i L\qquad \frac{x:A,\Gamma\vdash x:B}{\Gamma\vdash x:A\rightarrow_i B}\ \rightarrow_i R$$

$$\frac{x:A\rightarrow_c B,\Gamma\vdash x:A\quad x:B,\Gamma\vdash x:\bot}{x:A\rightarrow_c B,\Gamma\vdash x:\bot}\ \rightarrow_c L\qquad \frac{x:A,x:\neg B,\Gamma\vdash x:\bot}{\Gamma\vdash x:A\rightarrow_c B}\ \rightarrow_c R$$

$$\frac{x:\neg A,\Gamma\vdash x:A}{x:\neg A,\Gamma\vdash x:\bot}\ \neg L\qquad \frac{x:A,\Gamma\vdash x:\bot}{\Gamma\vdash x:\neg A}\ \neg R\qquad \frac{}{x:\bot,\Gamma\vdash z:C}\ \bot$$

$$\frac{\Gamma,x:P_i\vdash x:\bot}{\Gamma,x:P_c\vdash x:\bot}\ L_c\qquad \frac{\Gamma,x:\neg P_i\vdash x:\bot}{\Gamma\vdash x:P_c}\ R_c$$

Modal rules

$$\frac{xRy,y:A,x:\Box A,\Gamma\vdash z:C}{xRy,x:\Box A,\Gamma\vdash z:C}\ \Box L\qquad \frac{xRy,\Gamma\vdash y:A}{\Gamma\vdash x:\Box A}\ \Box R\qquad \frac{xRy,y:A,\Gamma\vdash z:C}{x:\Diamond_i A,\Gamma\vdash z:C}\ \Diamond_i L$$

$$\frac{xRy,\Gamma\vdash y:A}{xRy,\Gamma\vdash x:\Diamond_i A}\ \Diamond_i R\qquad \frac{xRy,y:A,\Gamma\vdash x:\bot}{x:\Diamond_c A,\Gamma\vdash x:\bot}\ \Diamond_c L\qquad \frac{x:\Box\neg A,\Gamma\vdash x:\bot}{\Gamma\vdash x:\Diamond_c A}\ \Diamond_c R$$

Fig. 2. Ecumenical modal system labEK. In rules $\Box R, \Diamond_i L, \Diamond_c L$, the eigenvariable y does not occur free in any formula of the conclusion.

$$\frac{xRy,\Gamma\vdash y:A}{\Gamma\vdash x:\Box A}\ \Box R\quad\rightsquigarrow\quad \frac{[xRy,\Gamma\vdash y:A]}{[\Gamma\vdash x:\Box A]}\ (\forall R,\rightarrow_i R)$$

$$\frac{xRy,y:A,\Gamma\vdash z:C}{x:\Diamond_i A,\Gamma\vdash z:C}\ \Diamond_i L\quad\rightsquigarrow\quad \frac{[xRy,y:A,\Gamma\vdash z:C]}{[x:\Diamond_i A,\Gamma\vdash z:C]}\ (\exists_i L,\wedge L)$$

$$\frac{xRy,y:A,\Gamma\vdash x:\bot}{x:\Diamond_c A,\Gamma\vdash x:\bot}\ \Diamond_c L\quad\rightsquigarrow\quad \frac{[xRy,y:A,\Gamma\vdash x:\bot]}{[x:\Diamond_c A,\Gamma\vdash x:\bot]}\ (\exists_c L,\wedge L)$$

$$\frac{xRy,\Gamma\vdash y:A}{xRy,\Gamma\vdash x:\Diamond_i A}\ \Diamond_i R\quad\rightsquigarrow\quad \frac{\overline{R(x,y),[\Gamma]\Rightarrow R(x,y)}\quad [xRy,\Gamma\vdash y:A]}{[xRy,\Gamma\vdash x:\Diamond_i A]}\ (\exists_i R,\wedge R)$$

$$\frac{x : \Box\neg A, \Gamma \vdash x : \bot}{\Gamma \vdash x : \Diamond_c A}\ \Diamond_c R \quad \rightsquigarrow \quad \frac{\forall y.\neg(R(x,y) \wedge [A]^e_y), [\Gamma] \Rightarrow \bot}{[\Gamma \vdash x : \Diamond_c A]}\ (\exists_c R)$$

(1) $\Rightarrow$ (2) is easily proved by induction on a proof of $\Gamma \vdash x : A$ in labEK, observing that $\vdash_{\mathsf{LEci}} \forall y.\neg(R(x,y) \wedge [A]^e_y) \leftrightarrow_i (\forall y.R(x,y) \rightarrow_i [\neg A]^e_y) = [\Box\neg A]^e_x$.

For proving (2) $\Rightarrow$ (1) observe that the rules $\rightarrow_i R, \wedge L, \wedge R$ are invertible in LEci and $\rightarrow_i L$ is semi-invertible on the right (*i.e.* if its conclusion is valid, so is its right premise). Hence, in the translated derivations in LEci *provability* is maintained from the end-sequent to the open leaves. This means that choosing a formula $[A]^e_x$ to work on is equivalent to performing all the steps of the translation given above, ending with translated sequents of smaller proofs. Therefore, any derivation of $[\Gamma \vdash x : A]$ in LEci can be transformed into a derivation of the same sequent where all the steps of the translation are actually performed. This is, in fact, one of the pillars of the *focusing* method [1,9]. In order to illustrate this, consider the derivation

$$\frac{R(x,y), R(x,y) \rightarrow_i \overset{\pi}{[A]^e_y}, [\Box A]^e_x, [\Gamma] \Rightarrow [C]^e_z}{R(x,y), [\Box A]^e_x, [\Gamma] \Rightarrow [C]^e_z}\ (\forall L)$$

where one decides to work on the formula $\forall y.(R(x,y) \rightarrow_i [A]^e_y) = [\Box A]^e_x$ obtaining a premise containing the formula $B = R(x,y) \rightarrow_i [A]^e_y$, with proof π. Since $\rightarrow_i L$ is semi-invertible on the right and the left premise is straightforwardly provable, then π can be substituted by the proof:

$$\frac{\overline{R(x,y), R(x,y) \rightarrow_i [A]^e_y, [\Box A]^e_x, [\Gamma] \Rightarrow R(x,y)} \quad \overset{\pi'}{R(x,y), [A]^e_y, [\Box A]^e_x, [\Gamma] \Rightarrow [C]^e_z}}{R(x,y), R(x,y) \rightarrow_i [A]^e_y, [\Box A]^e_x, [\Gamma] \Rightarrow [C]^e_z}\ \rightarrow_i L$$

where $\pi' = \pi$ if B is never principal in π, while π' is constructed from π by permuting down the application of the rule $\rightarrow_i L$ over B. Thus, by inductive hypothesis, $[xRy, y : A, x : \Box A, \Gamma \vdash z : C]$ is provable in labEK.

Finally, observe that, when restricted to the intuitionistic and neutral operators, labEK matches *exactly* Simpson's sequent system $\mathcal{L}_{\Box\Diamond}$ [20]. This implies that $\mathcal{L}_{\Box\Diamond}$ is trivially embedded into labEK. The analyticity of labEK is presented next.

4.1 Cut-elimination for labEK

In face of Theorem 2 most of the proof theoretical properties of the system labEK can be inherited from LEci. It is not different for the property of cut-elimination. Hence we will only illustrate the process here.

The extension of the Ecumenical weight for formulas presented [14] to modalities is defined bellow.

Definition 2. *The Ecumenical weight (*ew*) of a formula in $\mathcal{L}$ is recursively defined as*

– $\mathsf{ew}(P_i) = \mathsf{ew}(\bot) = 0$*;*

– $\mathsf{ew}(A \star B) = \mathsf{ew}(A) + \mathsf{ew}(B) + 1$ *if* $\star \in \{\wedge, \rightarrow_i, \vee_i\}$;
– $\mathsf{ew}(\heartsuit A) = \mathsf{ew}(A) + 1$ *if* $\heartsuit \in \{\neg, \Diamond_i, \Box\}$;
– $\mathsf{ew}(A \circ B) = \mathsf{ew}(A) + \mathsf{ew}(B) + 4$ *if* $\circ \in \{\rightarrow_c, \vee_c\}$;
– $\mathsf{ew}(P_c) = 4$;
– $\mathsf{ew}(\Diamond_c A) = \mathsf{ew}(A) + 4$.

Intuitively, the Ecumenical weight measures the amount of extra information needed (the negations added) in order to define the classical connectives from the intuitionistic and neutral ones.

Theorem 3. *The rule*

$$\frac{\Gamma \vdash x : A \quad x : A, \Gamma \vdash z : C}{\Gamma \vdash z : C}\ \mathsf{cut}$$

is admissible in labEK.

Proof. The proof is by the usual Gentzen method. The principal cases either eliminate the top-most cut or substitute it for cuts over simpler ecumenical formulas, *e.g.*

$$\dfrac{\dfrac{\overset{\pi_1}{x : \Box\neg A, \Gamma \vdash x : \bot}}{\Gamma \vdash x : \Diamond_c A}\ \Diamond_c R \quad \dfrac{\overset{\pi_2}{xRy, y : A, \Gamma \vdash x : \bot}}{x : \Diamond_c A, \Gamma \vdash z : \bot}\ \Diamond_c L}{\Gamma \vdash z : \bot}\ \mathsf{cut} \leadsto$$

$$\dfrac{\dfrac{\dfrac{\overset{\pi_2}{xRy, y : A, \Gamma \vdash y : \bot}}{xRy, \Gamma \vdash y : \neg A}\ \neg R}{\Gamma \vdash x : \Box\neg A}\ \Box R \quad \overset{\pi_1}{x : \Box\neg A, \Gamma, \vdash z : \bot}}{\Gamma \vdash z : \bot}\ \mathsf{cut}$$

Observe that the label of bottom is irrelevant due to the weakening rule W (that we have suppressed). Hence the Ecumenical weight on the cut formula passes from $\mathsf{ew}(\Diamond_c x.A) = \mathsf{ew}(A) + 4$ to $\mathsf{ew}(\Box\neg A) = \mathsf{ew}(A) + 2$.

The non-principal cuts can be flipped up as usual, generating cuts with smaller cut-height.

5 Axiomatization and semantics

Classical modal logic K is characterized as propositional classical logic, extended with the *necessitation rule* (presented in Hilbert style) $A/\Box A$ and the *distributivity axiom* $\mathsf{k} :\ \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$. Intuitionistic modal logic should then consist of propositional intuitionistic logic plus necessitation and distributivity.

As it is well known [16,20], there are many variants of axiom k that induces classically, but not intuitionistically, equivalent systems. In fact, the following axioms classically follow from k and the De Morgan laws, but not in an intuitionistic setting

$$\mathsf{k}_1 : \Box(A \to B) \to (\Diamond A \to \Diamond B) \qquad \mathsf{k}_2 : \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$$
$$\mathsf{k}_3 : (\Diamond A \to \Box B) \to \Box(A \to B) \qquad \mathsf{k}_4 : \Diamond\bot \to \bot$$

The combination of axiom k with axioms $\mathsf{k}_1 - \mathsf{k}_4$ characterizes intuitionistic modal logic IK [16]. And $\mathcal{L}_{\Box\Diamond}$ is sound and complete w.r.t. IK [20].

In the ecumenical setting, there are many more variants from k, depending on the classical or intuitionistic interpretation of the implication and diamond. Since $\mathcal{L}_{\Box\Diamond}$ is a sub-system of labEK, EK is *complete* w.r.t. the intuitionistic version of this set of axioms. *Soundness* will be established next.

For the sake of readability, we will abuse the notation and represent the connectives of $\mathsf{IK}/\mathcal{L}_{\Box\Diamond}$ using the neutral/intuitionistic correspondents in $\mathsf{EK}/\mathsf{labEK}$.

Definition 3. *Let* $[\![\cdot]\!]_{\mathsf{K}}$ *be the following formula translation from* EK *to* IK

$$\begin{array}{llll}
[\![p_i]\!]_{\mathsf{K}} & = p_i & [\![p_c]\!]_{\mathsf{K}} & = \neg(\neg(p_i)) \\
[\![\bot]\!]_{\mathsf{K}} & = \bot & [\![\neg A]\!]_{\mathsf{K}} & = \neg[\![A]\!]_{\mathsf{K}} \\
[\![A \wedge B]\!]_{\mathsf{K}} & = [\![A]\!]_{\mathsf{K}} \wedge [\![B]\!]_{\mathsf{K}} & [\![A \vee_i B]\!]_{\mathsf{K}} & = [\![A]\!]_{\mathsf{K}} \vee_i [\![B]\!]_{\mathsf{K}} \\
[\![A \to_i B]\!]_{\mathsf{K}} & = [\![A]\!]_{\mathsf{K}} \to_i [\![B]\!]_{\mathsf{K}} & [\![A \vee_c B]\!]_{\mathsf{K}} & = \neg(\neg[\![A]\!]_{\mathsf{K}} \wedge \neg[\![B]\!]_{\mathsf{K}}) \\
[\![A \to_c B]\!]_{\mathsf{K}} & = \neg([\![A]\!]_{\mathsf{K}} \wedge \neg[\![B]\!]_{\mathsf{K}}) & [\![\Box A]\!]_{\mathsf{K}} & = \Box[\![A]\!]_{\mathsf{K}} \\
[\![\Diamond_i A]\!]_{\mathsf{K}} & = \Diamond_i[\![A]\!]_{\mathsf{K}} & [\![\Diamond_c A]\!]_{\mathsf{K}} & = \neg\Box\neg[\![A]\!]_{\mathsf{K}}
\end{array}$$

The translation $[\![\cdot]\!] : \mathsf{labEK} \to \mathcal{L}_{\Box\Diamond}$ *is defined as* $[\![x : A]\!] = x : [\![A]\!]_{\mathsf{K}}$ *and assumed identical on relational atoms.*

Since the translations above preserve the double-negation interpretation of classical connectives into intuitionistic (modal) logic, the following holds.

Lemma 1. $\vdash_{\mathsf{labEK}} \Gamma \vdash x : A$ *iff* $\vdash_{\mathsf{labEK}} [\![\Gamma \vdash x : A]\!]$ *iff* $\vdash_{\mathcal{L}_{\Box\Diamond}} [\![\Gamma \vdash x : A]\!]$.

Hence we have that

Theorem 4. EK *is sound w.r.t. the intuitionistic version of axioms* $\mathsf{k} - \mathsf{k}_4$*.*

Remark 1. One could ask: what happens if we exchange the intuitionistic versions of the connectives with classical ones? For answering that, consider $\mathsf{k}_{\alpha\beta\gamma} : \Box(A \to_\alpha B) \to_\beta (\Box A \to_\gamma \Box B)$ with $\alpha, \beta, \gamma \in \{i, c\}$. Since $C \to_i D \Rightarrow C \to_c D$ in EK, $\mathsf{k}_{\alpha ii} \Rightarrow \mathsf{k}_{\alpha\beta\gamma}$ for any value of β, γ. Moreover, k_{cii} is not provable in EK. Hence, k_{iii} is the *minimal* version of k provable in EK. The same holds for all the other axioms.

5.1 Ecumenical birelational models

In [2], the negative translation was used to relate cut-elimination theorems for classical and intuitionistic logics. Since part of the argumentation was given semantically, a notion of Kripke semantics for classical logic was stated, via the respective semantics for intuitionistic logic and the double negation interpretation (see also [7]). In [14] a similar definition was given, but under the ecumenical approach, and it was extended to the first-order case in [15]. We will propose a birelational Kripke semantics for ecumenical modal logic, which is an extension of the proposal in [14] to modalities.

Definition 4. *A* birelational Kripke model *is a quadruple* $\mathcal{M} = (W, \leq, R, V)$ *where* (W, R, V) *is a Kripke model such that* W *is partially ordered with order* $\leq$, *the satisfaction function* $V : \langle W, \leq\rangle \to \langle 2^{\mathcal{P}}, \subseteq\rangle$ *is monotone and:*
F1. For all worlds w, v, v', *if* wRv *and* $v \leq v'$, *there is a* w' *such that* $w \leq w'$ *and* $w'Rv'$;
F2. For all worlds w', w, v, *if* $w \leq w'$ *and* wRv, *there is a* v' *such that* $w'Rv'$ *and* $v \leq v'$.

An ecumenical modal Kripke model *is a birelational Kripke model such that truth of an ecumenical formula at a point* w *is the smallest relation* $\models_{\mathsf{E}}$ *satisfying*

$\mathcal{M}, w \models_{\mathsf{E}} p_i$	*iff*	$p_i \in V(w)$;
$\mathcal{M}, w \models_{\mathsf{E}} A \wedge B$	*iff*	$\mathcal{M}, w \models_{\mathsf{E}} A$ *and* $\mathcal{M}, w \models_{\mathsf{E}} B$;
$\mathcal{M}, w \models_{\mathsf{E}} A \vee_i B$	*iff*	$\mathcal{M}, w \models_{\mathsf{E}} A$ *or* $\mathcal{M}, w \models_{\mathsf{E}} B$;
$\mathcal{M}, w \models_{\mathsf{E}} A \to_i B$	*iff*	*for all* v *such that* $w \leq v, \mathcal{M}, v \models_{\mathsf{E}} A$ *implies* $\mathcal{M}, v \models_{\mathsf{E}} B$;
$\mathcal{M}, w \models_{\mathsf{E}} \neg A$	*iff*	*for all* v *such that* $w \leq v, \mathcal{M}, v \not\models_{\mathsf{E}} A$;
$\mathcal{M}, w \models_{\mathsf{E}} \bot$		*never holds*;
$\mathcal{M}, w \models_{\mathsf{E}} \Box A$	*iff*	*for all* v, w' *such that* $w \leq w'$ *and* $w'Rv, \mathcal{M}, v \models_{\mathsf{E}} A$.
$\mathcal{M}, w \models_{\mathsf{E}} \Diamond_i A$	*iff*	*there exists* v *such that* wRv *and* $\mathcal{M}, v \models_{\mathsf{E}} A$.
$\mathcal{M}, w \models_{\mathsf{E}} p_c$	*iff*	$\mathcal{M}, w \models_{\mathsf{E}} \neg(\neg p_i)$;
$\mathcal{M}, w \models_{\mathsf{E}} A \vee_c B$	*iff*	$\mathcal{M}, w \models_{\mathsf{E}} \neg(\neg A \wedge \neg B)$;
$\mathcal{M}, w \models_{\mathsf{E}} A \to_c B$	*iff*	$\mathcal{M}, w \models_{\mathsf{E}} \neg(A \wedge \neg B)$.
$\mathcal{M}, w \models_{\mathsf{E}} \Diamond_c A$	*iff*	$\mathcal{M}, w \models_{\mathsf{E}} \neg\Box\neg A$.

Since, restricted to intuitionistic and neutral connectives, $\models_{\mathsf{E}}$ is the usual birelational interpretation $\models$ for IK (and, consequently, $\mathcal{L}_{\Box\Diamond}$ [20]), and since the classical connectives are interpreted via the neutral ones using the double-negation translation, an ecumenical modal Kripke model is nothing else than the standard birelational Kripke model for intuitionistic modal logic IK. Hence, in the face of Theorem 4, the following result is trivial.

Theorem 5. EK *is sound and complete w.r.t. the ecumenical modal Kripke semantics, that is,* $\vdash_{\mathsf{EK}} A$ *iff* $\models_{\mathsf{E}} A$.

6 Extensions

Depending on the application, several further modal logics can be defined as extensions of K by simply restricting the class of frames we consider. Many of the restrictions one can be interested in are definable as formulas of first-order logic, where the binary predicate $R(x, y)$ refers to the corresponding accessibility relation. Table 1 summarizes some of the most common logics, the corresponding frame property, together with the modal axiom capturing it in the intuitionistic framework [18].

We divide the problem of modal extensions in 3 parts: (i) transform the FO formulas in Table 1 into inference rules; (ii) prove that the axioms are theorems in the extended systems; and (iii) prove that the axioms actually enforce the respective condition. In this work, we will show (i) and (ii), and start the discussion of (iii) in the ecumenical setting.

Axiom	**Condition**	**First-Order (FO) Formula**
$\mathsf{T}: \Box A \to A \wedge A \to \Diamond A$	Reflexivity	$\forall x. R(x,x)$
$\mathsf{4}: \Box A \to \Box\Box A \wedge \Diamond\Diamond A \to \Diamond A$	Transitivity	$\forall x,y,z.(R(x,y) \wedge R(y,z)) \to R(x,z)$
$\mathsf{5}: \Box A \to \Box\Diamond A \wedge \Diamond\Box A \to \Diamond A$	Euclideaness	$\forall x,y,z.(R(x,y) \wedge R(x,z)) \to R(y,z)$
$\mathsf{B}: A \to \Box\Diamond A \wedge \Diamond\Box A \to A$	Symmetry	$\forall x,y. R(x,y) \to R(y,x)$

Table 1. Axioms and corresponding first-order conditions on R.

$$\frac{xRx, \Gamma \vdash w : C}{\Gamma \vdash w : C}\ \mathsf{T} \qquad \frac{xRz, xRy, yRz, \Gamma \vdash w : C}{xRy, yRz, \Gamma \vdash w : C}\ \mathsf{4}$$

$$\frac{yRz, xRy, xRz, \Gamma \vdash w : C}{xRy, xRz, \Gamma \vdash w : C}\ \mathsf{5} \qquad \frac{yRx, xRy, \Gamma \vdash w : C}{xRy, \Gamma \vdash w : C}\ \mathsf{B}$$

Fig. 3. Labeled sequent rules corresponding to axioms in Table 1.

The general problem of systematically extending standard proof-theoretical results obtained for pure logic to non-logical axioms has been focus of attention for quite some time now (see *e.g.* [20,13,22,6,4]). In [11], a systematic procedure for transforming a class of FO formulas (called *bipolars*) into rules for atoms was presented. This procedure involves polarization of formulas and focusing [1].

The main idea of this method is that assuming a FO clause as a theory is the same as applying a rule, determined by the shape of the formula, on its atomic subformulas. In order to illustrate the process, consider the formula $\forall x, y. R(x,y) \to_i R(y,x)$. Having it as a theory and working throughout it in LEci:

$$\dfrac{\dfrac{\Gamma \vdash R(x,y) \quad \Gamma, R(y,x) \vdash C}{\Gamma, R(x,y) \to_i R(y,x) \vdash C}\ \to_i L}{\Gamma, \forall x,y. R(x,y) \to_i R(y,x) \vdash C}\ \forall L$$

From that, there are two protocols: the *forward-chaining* insists that the left premise above is trivial, meaning that it is proved by the initial rule (hence $R(x,y) \in \Gamma$ and $R(y,x)$ is "produced" from it); and the *backward-chaining* insists that the right-most premise is trivial: that is, $R(y,x)$ and C are the same atomic formula. These protocols give rise to the rules, respectively

$$\frac{\Gamma, R(y,x) \vdash C}{\Gamma, R(x,y) \vdash C} \qquad \frac{\Gamma \vdash R(x,y)}{\Gamma \vdash R(y,x)}$$

The first one appears, *e.g.* in [20], while the second occurs in [22]. Adopting the forward-chaining protocol, the FO formulas in Table 1 are transformed into the rules in Fig. 3, hence fulfilling the goal (i). Observe that, as noted in Remark 1, we only need to consider the intuitionistic version of the FO formulas.

Regarding (ii), it is easy to see that the axioms listed in the first column of Table 1 are theorems in labEK extended with the respective rules. For example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{xRx, x : A \vdash x : A}\,\mathsf{init}}{xRx, x : \Box A \vdash x : A}\,\Box L}{x : \Box A \vdash x : A}\,\mathsf{T}}{\vdash x : \Box A \to_i A}\,\to_i R \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{xRx, x : A \vdash x : A}\,\mathsf{init}}{xRx, x : A \vdash x : \Diamond_i A}\,\Diamond_i R}{x : A \vdash \Diamond_i A}\,\mathsf{T}}{\vdash x : A \to_i \Diamond_i A}\,\to_i R}{\vdash x : (\Box A \to_i A) \wedge (A \to_i \Diamond_i A)}\,\wedge R$$

Finally, item (iii) remains: do such axioms indeed reflect the respective conditions in Fig. 3? We illustrate the complexity of the interaction of modal axioms and ecumenical connectives in the case of axiom T.

It is well known that, by itself, $\Box A \to_i A$ does not enforce reflexivity of an intuitionistic model [20,21]. In fact, the derivation above shows that, in frames having the reflexivity property, both $\Box A \to_i A$ and $A \to_i \Diamond_i A$ are provable. For the converse, since $\Box$ and $\Diamond_i$ are not inter-definable, we need to add $A \to_i \Diamond_i A$ in order to still be complete w.r.t. reflexive models. This is also true for any extension of EK by path axioms plus contrapositives w.r.t. their corresponding models. But beyond that it wouldn't be as clean cut, unless one adds the preorder relation into the mix as in [16]. This not mentioning the ecumenical nature of *atoms* [10]. In fact, in the ecumenical setting, the possibility of mixing intuitionistic and classical relational formulas and modalities can make this discussion even harder, and it is left for a future work.

Finally, since adding ecumenical axioms can deeply affect the resulting system, it should be studied carefully. For example, the modalities in EK are not inter-definable, but what would be the consequence of adding $\neg\Diamond_i\neg A \to_i \Box A$ as an extra axiom to labEK added with the rule T? The following derivation shows that the addition of this new axiom has a disastrous propositional consequence.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{xRy, y : A, y : \neg(A \vee_i \neg A) \vdash y : A}\,\mathsf{init}}{xRy, y : A, y : \neg(A \vee_i \neg A) \vdash y : \bot}\,\neg L, \vee_i R1}{xRy, y : \neg(A \vee_i \neg A) \vdash x : \bot}\,\neg L, \vee_i R2, \neg R}{x : \Diamond_i\neg(A \vee_i \neg A) \vdash x : \bot}\,\Diamond_i L}{\vdash x : \neg\Diamond_i\neg(A \vee_i \neg A)}\,\neg R}{\vdash x : \Box(A \vee_i \neg A)}\,eq \qquad \dfrac{\dfrac{\dfrac{}{xRx, x : (A \vee_i \neg A) \vdash x : (A \vee_i \neg A)}\,\mathsf{init}}{xRx, x : \Box(A \vee_i \neg A) \vdash x : (A \vee_i \neg A)}\,\Box L}{x : \Box(A \vee_i \neg A) \vdash x : (A \vee_i \neg A)}\,\mathsf{T}}{\vdash x : (A \vee_i \neg A)}\,\mathsf{cut}$$

where eq represents the proof steps of the substitution of a boxed formula for its diamond version.[5] That is, if $\Box$ and $\Diamond_i$ are inter-definable, then $A \vee_i \neg A$ is a theorem and intuitionistic KT collapses to a classical system!

[5] We have presented a proof with cut for clarity, remember that labEK has the cut-elimination property (see Appendix 4.1).

7 Discussion and conclusion

Some questions naturally arise with respect to ecumenical systems: what (really) are ecumenical systems? What are they good for? Why should anyone be interested in ecumenical systems? What is the real motivation behind the definition and development of ecumenical systems? Based on the specific case of the ecumenical system that puts classical logic and intuitionist logic coexisting in peace in the same codification, we would like to propose three possible motivations for the definition, study and development of ecumenical systems.

Philosophical motivation This was the motivation of Prawitz. Inferentialism, and in particular, logical inferentialism, is the semantical approach according to which the meaning of the logical constants can be specified by the rules that determine their correct use. According to Prawitz [17],

> *"Gentzen's introduction rules, taken as meaning constitutive of the logical constants of the language of predicate logic, agree, as is well known, with how intuitionistic mathematicians use the constants. On the one hand, the elimination rules stated by Gentzen become all justified when the constants are so understood because of there being reductions, originally introduced in the process of normalizing natural deductions, which applied to proofs terminating with an application of elimination rules give canonical proofs of the conclusion in question. On the other hand, no canonical proof of an arbitrarily chosen instance of the law of the excluded middle is known, nor any reduction that applied to a proof terminating with an application of the classical form of reductio ad absurdum gives a canonical proof of the conclusion."*

But what about the use classical mathematicians make of the logical constants? Again, according to Prawitz,

> *"What is then to be said about the negative thesis that no coherent meaning can be attached on the classical use of the logical constants? Gentzen's introduction rules are of course accepted also in classical reasoning, but some of them cannot be seen as introduction rules, that is they cannot serve as explanations of meaning. The classical understanding of disjunction is not such that $A \vee B$ may be rightly asserted only if it is possible to prove either A or B, and hence Gentzen's introduction rule for disjunction does not determine the meaning of classical disjunction."*

As an alternative, in a recent paper [12] Murzi presents a different approach to the extension of inferentialism to classical logic. There are some natural (proof-theoretical) inferentialist requirements on admissible logical rules, such as harmony and separability (although harmonic, Prawitz' rules for the classical operators do not satisfy separability). According to Murzi, our usual logical practice does not seem to allow for an inferentialist account of classical logic (unlike what happens with respect to intuitionistic logic). Murzi proposes a new set of rules for classical logical operators based on: absurdity as a punctuation mark, and Higher-level rules [19]. This allows for a "pure" logical system, where negation is not used in premises.

Mathematical/computational motivation (This was actually the original motivation for proposing ecumenical systems.) The first ecumenical system (as far as we know) was defined by Krauss in a technical report of the University of Kassel [8] (the text was never published in a journal). The paper is divided in two parts: in the first part, Krauss' ecumenical system is defined and some properties proved. In the second part, some theorems of basic algebraic number theory are revised in the light of this (ecumenical) system, where constructive proofs of some "familiar classical proofs" are given (like the proof of Dirichlet's Unit Theorem). The same motivation can be found in the final passages of the paper [5], where Dowek examines what would happen in the case of axiomatizations of mathematics. Dowek gives a simple example from Set Theory, and ends the paper with this very interesting remark:

> *"Which mathematical results have a classical formulation that can be proved from the axioms of constructive set theory or constructive type theory and which require a classical formulation of these axioms and a classical notion of entailment remains to be investigated."*

Logical motivation In a certain sense, the logical motivation naturally combines certain aspects of the philosophical motivation with certain aspects of the mathematical motivation. According to Prawitz, one can consider the so-called classical first order logic as *"an attempted codification of a fragment of inferences occurring in [our] actual deductive practice"*. Given that there exist different and even divergent attempts to codify our (informal) deductive practice, it is more than natural to ask about what relations are entertained between these codifications. Ecumenical systems may help us to have a better understanding of the relation between classical logic and intuitionistic logic. But one could say that, from a logical point of view, there's nothing new in the ecumenical proposal: Based on translations, the new classical operators could be easily introduced by "explicit definitions". Let us consider the following dialogue between a classical logician (CL) and an intuitionistic logician (IL), a dialogue that may arise as a consequence of the translations mentioned above:

- IL: if what you mean by $(A \vee B)$ is $\neg(\neg A \wedge \neg B)$, then I can accept the validity of $(A \vee \neg A)$!
- CL: but I do not mean $\neg(\neg A \wedge \neg\neg A)$ by $(A \vee \neg A)$. One must distinguish the excluded-middle from the the principle of non-contradiction. When I say that Goldbach's conjecture is either true or false, I am not saying that it would be contradictory to assert that it is not true and that it is not the case that it is not true!
- IL: but you must realize that, at the end of the day, you just have one logical operator, the Sheffer stroke (or the Quine's dagger).

- CL: But this is not at all true! The fact that we can define one operator in terms of other operators does not imply that we don't have different operators! We do have 16 binary propositional operators (functions). It is also true that we can prove $\vdash (A \vee_c B) \leftrightarrow \neg(\neg A \wedge \neg B)$ in the ecumenical system, but this does not mean that we don't have three different operators, $\neg$, $\vee_c$ and $\wedge$.

Maybe we can resume the logical motivation in the following (very simple) sentence:

> Ecumenical systems constitute a new and promising instrument to study the nature of different (maybe divergent!) logics.

Now, what can we say about *modal* ecumenical systems? Regarding the **philosophical view**, in [15] we have used invertibility results in order to obtain a sequent system for Prawitz' ecumenical logic with a minimal occurrences of negations, moving then towards a "purer" ecumenical system. Nevertheless, negation still plays an important role on interpreting classical connectives. This is transferred to our definition of ecumenical modalities, where the classical possibility is interpreted using negation. We plan to investigate what would be the meaning of classical possibility without impure rules. For the **mathematical view**, our use of intuitionistic/classical/neutral connectives allows for a more chirurgical detection of the parts of a mathematical proof that are intrinsically intuitionistic, classical or independent. We now bring this discussion to modalities. There is an interesting aspect of this expansion, that would be the ecumenical interpretation of *relational formulas*, as noted in Section 6. Finally, concerning the **logical view**, it would be interesting to explore some relations between general results on translations and ecumenical systems.

We end the present text by noting that there is an obvious connection between the Ecumenical approach and Gödel-Gentzen's double-negation translations of classical logic into intuitionistic logic. This could lead to the erroneous conclusion that the ecumenical refinement of classical logic is *essentially* the same refinement produced by such translation. But, on a closer inspection, shows that this is not true! Indeed, classical mathematical practice does not require that every occurrence of $\vee$ in real mathematical proofs be replaced by its Gödel-Gentzen translation. For example, there is no reason to translate the occurrence of $\vee$ in the theorem $(A \rightarrow (A \vee B))$. Given that the Gödel-Gentzen translation function systematically and globally eliminates every occurrence of $\vee$ and $\exists$ from the language of classical logic, one may say that the ecumenical system reflects more faithfully the "local" necessary uses of classical reasoning.

That is, the ecumenical refinement "interpolates" the Gödel-Gentzen-translation function. And this is extended, in our work, to reasoning with modalities.

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