Decomposing labelled proof theory for intuitionistic modal logic

Sonia Marin*
IT-Universitetet i København
Denmark

Marianela Morales
Universidad Nacional de Córdoba
Argentina

Lutz Straßburger
Inria Saclay & LIX
France

ACM Reference Format:
Sonia Marin, Marianela Morales, and Lutz Straßburger. 2018. Decomposing labelled proof theory for intuitionistic modal logic. In *Proceedings of the Second Women in Logic Workshop (WiL'18)*. ACM, New York, NY, USA, 1 page. https://doi.org/

Structural proof theoretic accounts of intuitionistic modal logic can adopt the paradigm of *labelled deduction* in the form of labelled natural deduction and sequent systems [3], or the one of *unlabelled deduction* in the form of sequent [1] or nested sequent systems [7] (for a survey see [4, Chap. 3]).

Simpson's labelled sequents make use only of relational atoms referring to the accessibility relation of a Kripke model. In this short note we propose a system that represents both the *accessibility relation* (for modal logics) and the *preorder relation* (for intuitionistic logic), using the full power of the bi-relational semantics for intuitionistic modal logics [5, 6], and developing fully the idea of [2].

A *bi-relational frame* [5, 6] $\mathcal{B}$ is a triple $\langle W, R, \leq\rangle$ of a non-empty set of worlds W equipped with an accessibility relation R and a preorder $\leq$, satisfying:

(F_1) For all worlds x, y, z, if xRy and $y \leq z$, there exists a u such that $x \leq u$ and uRz.

(F_2) For all worlds x, y, z, if xRy and $x \leq z$, there exists a u such that $y \leq u$ and zRu.

Reflecting this definition, we define our two-sided intuitionistic labelled sequents, similarly to [2], to be of the form $\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}$ with $\mathcal{B}$ a set of relational atoms xRy and preorder atoms $x \leq y$, and $\mathcal{L}, \mathcal{R}$ multi-sets of labelled formulas $x : A$ (for x, y labels and A an intuitionistic modal formula).

Furthermore, our system has to incorporate the two semantic conditions into deductive rules as follows:

$$F_1\ \frac{\mathcal{B}, xRy, y \leq z, x \leq u, uRz, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, xRy, y \leq z, \mathcal{L} \Rightarrow \mathcal{R}}\ u \text{ fresh}$$

$$F_2\ \frac{\mathcal{B}, xRy, x \leq z, y \leq u, zRu, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, xRy, x \leq z, \mathcal{L} \Rightarrow \mathcal{R}}\ u \text{ fresh}$$

In the intuitionistic setting, the validity of a modal formula has to be defined using both the R and the $\leq$ relation as: $x \Vdash \Box A$ iff for all y and z s.t. $x \leq y$ and yRz, $z \Vdash A$.

*This publication was made possible in part by NPRP Grant #7-988-1-178 from the Qatar National Research Fund (a member of Qatar Foundation). The statements made herein are solely the responsibility of the authors.

WiL'18, 2018
2018.

Again, our system reflects exactly this definition in the rules introducing the $\Box$-operator:

$$\Box_L\ \frac{\mathcal{B}, x \leq y, yRz, \mathcal{L}, x : \Box A, z : A \Rightarrow \mathcal{R}}{\mathcal{B}, \mathcal{L}, x \leq y, yRz, x : \Box A \Rightarrow \mathcal{R}}$$

$$\Box_R\ \frac{\mathcal{B}, x \leq y, yRz, \mathcal{L} \Rightarrow \mathcal{R}, z : A}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x : \Box A}\ y, z \text{ fresh}$$

By complementing these rules with the standard labelled rules for intuitionistic modal logic of [3], we get a system that is sound and complete wrt. the birelational semantics.

In [6], Plotkin and Stirling give a correspondence result for intuitionistic modal logic extended with a family of axioms wrt. some classes of bi-relational frames. For example, the frames that validate the axiom $4_\Diamond : \Diamond\Diamond A \supset \Diamond A$ are exactly the ones satisfying the condition:

($\Diamond\!\!\!+_4$) if wRv and vRu, there exists a u' s.t. $u \leq u'$ and wRu'.

Incorporating the preorder symbol into the syntax of our sequents allows us to also obtain a sound and complete proof system for the intuitionistic modal logic extended with axiom $4_\Diamond$, by designing the following rule:

$$\Diamond\!\!\!+_4\ \frac{\mathcal{B}, wRv, vRu, u \leq u', wRu', \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, wRv, vRu, \mathcal{L} \Rightarrow \mathcal{R}}\ u' \text{ fresh}$$

Therefore, we decompose further the formalism of labelled sequents and extend the reach of labelled deduction to the logics studied in [6]. These systems enjoy cut-elimination via usual arguments, the generality of the result is subject of ongoing study.

References

[1] G. M. Bierman and V. de Paiva. *On an Intuitionistic Modal Logic* Studia Logica, 2000.
[2] P. Maffezioli, A. Naibo, and S. Negri. *The Church-Fitch knowability paradox in the light of structural proof theory.* Synthese, 2013.
[3] A. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logic.* PhD thesis, University of Edinburgh, 1994.
[4] S. Marin. *Modal proof theory through a focused telescope.* PhD thesis, Université Paris-Saclay, 2018.
[5] G. Fischer-Servi. *Axiomatizations for some intuitionistic modal logics.* R. del Seminario Matematico della Univ. Politecnica di Torino, 1984.
[6] G. D. Plotkin and C. P. Stirling. *A framework for intuitionistic modal logic.* In J. Y. Halpern, editor, 1st Conference on Theoretical Aspects of Reasoning About Knowledge. Morgan Kaufmann, 1986.
[7] L. Straßburger. *Cut Elimination in Nested Sequents for Intuitionistic Modal Logics.* In F. Pfenning, editor, 16th Conference on Foundations of Software Science and Computation Structures. Springer, 2013.