Label-free Modular Systems for Classical and Intuitionistic Modal Logics

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Classical Modal Logic

- Formulas:

$$A, B, \ldots ::= p \mid \bar{p} \mid A \wedge B \mid A \vee B \mid \Box A \mid \Diamond A$$

- Negation: De Morgan laws and $\overline{\Box A} = \Diamond \bar{A}$

- Axioms for K: classical propositional logic and

$$\mathsf{k}: \quad \Box(A \supset B) \supset (\Box A \supset \Box B)$$

- Rules: modus ponens: $\dfrac{A \quad A \supset B}{B}$ necessitation: $\dfrac{A}{\Box A}$

Intuitionistic Modal Logic

- Formulas:

$$A, B, \ldots ::= p \mid \bot \mid A \wedge B \mid A \vee B \mid A \supset B \mid \Box A \mid \Diamond A$$

- Negation: $\neg A = A \supset \bot$ and independance of the modalities

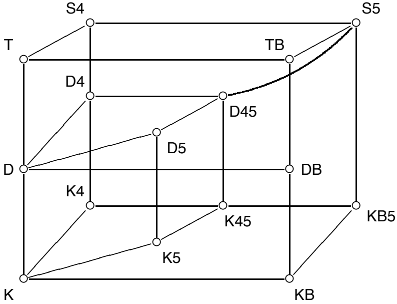
- Axioms for IK: intuitionistic propositional logic and

$$\begin{array}{ll}
\mathsf{k}_1: & \Box(A \supset B) \supset (\Box A \supset \Box B) \\
\mathsf{k}_2: & \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \\
\mathsf{k}_3: & \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) \\
\mathsf{k}_4: & (\Diamond A \supset \Box B) \supset \Box(A \supset B) \\
\mathsf{k}_5: & \neg\Diamond\bot
\end{array}$$

- Rules: modus ponens: $\dfrac{A \quad A \supset B}{B}$ necessitation: $\dfrac{A}{\Box A}$

Classical Modal Axioms

d : $\Box A \supset \Diamond A$
t : $A \supset \Diamond A$
b : $A \supset \Box\Diamond A$
4 : $\Diamond\Diamond A \supset \Diamond A$
5 : $\Diamond A \supset \Box\Diamond A$



Intuitionistic Modal Axioms

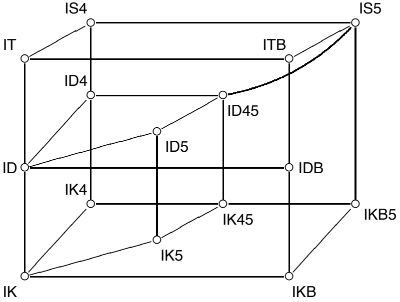
d: $\Box A \supset \Diamond A$

t: $A \supset \Diamond A \quad \wedge \quad \Box A \supset A$

b: $A \supset \Box\Diamond A \quad \wedge \quad \Diamond\Box A \supset A$

4: $\Diamond\Diamond A \supset \Diamond A \quad \wedge \quad \Box A \supset \Box\Box A$

5: $\Diamond A \supset \Box\Diamond A \quad \wedge \quad \Diamond\Box A \supset \Box A$



Nested Sequents for classical modal logic

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- Sequent:

$$\Gamma ::= A_1, \ldots, A_m$$

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- Corresponding formula:

$$fm(\Gamma) = A_1 \vee \ldots \vee A_m$$

Nested Sequents for classical modal logic

- Nested Sequent:

$$\Gamma ::= A_1, \ldots, A_m, [\Gamma_1], \ldots, [\Gamma_n]$$

- Corresponding formula:

$$fm(\Gamma) = A_1 \vee \ldots \vee A_m \vee \Box fm(\Gamma_1) \vee \ldots \vee \Box fm(\Gamma_n)$$

Nested Sequents for classical modal logic

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$$\Gamma ::= A_1, \ldots, A_m, [\Gamma_1], \ldots, [\Gamma_n]$$

- Corresponding formula:

$$fm(\Gamma) = A_1 \vee \ldots \vee A_m \vee \Box fm(\Gamma_1) \vee \ldots \vee \Box fm(\Gamma_n)$$

- A context is a sequent with one or several holes:

$$\Gamma\{\ \}\{\ \} = A, [B, \{\ \}, [\{\ \}], C]$$

Nested Sequents for classical modal logic

- Nested Sequent:

$$\Gamma ::= A_1, \ldots, A_m, [\Gamma_1], \ldots, [\Gamma_n]$$

- Corresponding formula:

$$fm(\Gamma) = A_1 \vee \ldots \vee A_m \vee \Box fm(\Gamma_1) \vee \ldots \vee \Box fm(\Gamma_n)$$

- A context is a sequent with one or several holes:

$$\Gamma\{\ \}\{\ \} = A, [B, \{\ \}, [\{\ \}], C]$$

$$\Gamma\{[D]\}\{A, [C]\} = A, [B, [D], [A, [C]], C]$$

Nested Sequents for intuitionistic modal logic

- Sequent:

$$\Gamma ::= A_1, \ldots, A_m \vdash B$$

- Corresponding formula:

$$A_1 \wedge \ldots \wedge A_m \supset B$$

Nested Sequents for intuitionistic modal logic

- Sequent:

$$\Gamma ::= A_1^{\bullet}, \ldots, A_m^{\bullet}, B^{\circ}$$

- Corresponding formula:

$$A_1 \wedge \ldots \wedge A_m \supset B$$

Nested Sequents for intuitionistic modal logic

- Nested Sequent:

$$\Gamma ::= \Lambda^{\bullet}, \Pi^{\circ}$$

- Corresponding formula:

$$\mathit{fm}(\Gamma) = \mathit{fm}(\Lambda^{\bullet}) \supset \mathit{fm}(\Pi^{\circ})$$

Nested Sequents for intuitionistic modal logic

- Nested Sequent:

$$\Gamma ::= \Lambda^{\bullet}, \Pi^{\circ}$$

$$\Lambda^{\bullet} ::= A_1^{\bullet}, ..., A_m^{\bullet}, [\Lambda_1^{\bullet}], ..., [\Lambda_n^{\bullet}]$$

- Corresponding formula:

$$fm(\Gamma) = fm(\Lambda^{\bullet}) \supset fm(\Pi^{\circ})$$

$$fm(\Lambda^{\bullet}) = A_1 \wedge ... \wedge A_m \wedge \Diamond fm(\Lambda_1^{\bullet}) \wedge ... \wedge \Diamond fm(\Lambda_n^{\bullet})$$

Nested Sequents for intuitionistic modal logic

- Nested Sequent:

$$\Gamma ::= \Lambda^{\bullet}, \Pi^{\circ}$$

$$\Lambda^{\bullet} ::= A_1^{\bullet}, ..., A_m^{\bullet}, [\Lambda_1^{\bullet}], ..., [\Lambda_n^{\bullet}]$$

$$\Pi^{\circ} ::= A^{\circ} \mid [\Gamma]$$

- Corresponding formula:

$$fm(\Gamma) = fm(\Lambda^{\bullet}) \supset fm(\Pi^{\circ})$$

$$fm(\Lambda^{\bullet}) = A_1 \wedge ... \wedge A_m \wedge \Diamond fm(\Lambda_1^{\bullet}) \wedge ... \wedge \Diamond fm(\Lambda_n^{\bullet})$$

$$fm([\Gamma]) = \Box fm(\Gamma)$$

Nested Sequent for intuitionistic modal logic

- Output context:

$$\Gamma_1\{\ \} = A^\bullet, [B^\bullet, \{\ \}]$$

Nested Sequent for intuitionistic modal logic

- Output context:

$$\Gamma_1\{\ \} = A^\bullet, [B^\bullet, \{\ \}]$$

$$\to \Gamma_1\{[C^\bullet, D^\circ]\} = A^\bullet, [B^\bullet, [C^\bullet, D^\circ]]$$

Nested Sequent for intuitionistic modal logic

- Output context:

$$\Gamma_1\{\ \} = A^\bullet, [B^\bullet, \{\ \}]$$

$$\rightarrow \Gamma_1\{[C^\bullet, D^\circ]\} = A^\bullet, [B^\bullet, [C^\bullet, D^\circ]]$$

- Input context

$$\Gamma_2\{\ \} = A^\bullet, [B^\circ, \{\ \}]$$

Nested Sequent for intuitionistic modal logic

- Output context:

$$\Gamma_1\{\ \} = A^\bullet, [B^\bullet, \{\ \}]$$

$$\to \Gamma_1\{[C^\bullet, D^\circ]\} = A^\bullet, [B^\bullet, [C^\bullet, D^\circ]]$$

- Input context

$$\Gamma_2\{\ \} = A^\bullet, [B^\circ, \{\ \}]$$

$$\to \Gamma_2\{[C^\bullet, D^\bullet]\} = A^\bullet, [B^\circ, [C^\bullet, D^\bullet]]$$

Classical Rules

System NK

$$\text{id}\ \frac{}{\Gamma\{a, \bar{a}\}} \qquad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}$$

$$\text{c}\ \frac{\Gamma\{A, A\}}{\Gamma\{A\}} \qquad \Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\ \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

Additional structural rules

$$\text{w}\ \frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \text{cut}\ \frac{\Gamma\{\bar{A}\} \quad \Gamma\{A\}}{\Gamma\{\emptyset\}}$$

Classical Rules

Modal $\Diamond$-rules

$$\mathsf{d}^{\Diamond}\ \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{t}^{\Diamond}\ \frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{b}^{\Diamond}\ \frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Diamond A]\}}$$

$$4^{\Diamond}\ \frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad 5^{\Diamond}\ \frac{\Gamma\{\emptyset\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) \geq 1$$

Modal structural rules

$$\mathsf{d}^{[]}\ \frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \qquad \mathsf{t}^{[]}\ \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}} \qquad \mathsf{b}^{[]}\ \frac{\Gamma\{[\Sigma, [\Delta]]\}}{\Gamma\{[\Sigma], \Delta\}}$$

$$4^{[]}\ \frac{\Gamma\{[\Delta], [\Sigma]\}}{\Gamma\{[[\Delta], \Sigma]\}} \qquad 5^{[]}\ \frac{\Gamma\{[\Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Delta]\}}\ \mathit{depth}(\Gamma\{\ \}\{[\Delta]\}) \geq 1$$

Classical Rules: Example

$$5 : \Diamond A \supset \Box \Diamond A$$

Classical Rules: Example

$$\vee \frac{\Box \bar{A}, \Box \Diamond A}{5 : \Diamond A \supset \Box \Diamond A}$$

Classical Rules: Example

$$\vee\ \dfrac{\Box\ \dfrac{\Box\bar{A}, [\Diamond A]}{\Box\bar{A}, \Box\Diamond A}}{5 : \Diamond A \supset \Box\Diamond A}$$

Classical Rules: Example

$$
\text{cut}\ \dfrac{\square\bar{A}, [\Diamond\Diamond A, \Diamond A] \quad \square\bar{A}, [\square\square\bar{A}, \Diamond A]}{\vee\ \dfrac{\square\ \dfrac{\square\bar{A}, [\Diamond A]}{\square\bar{A}, \square\Diamond A}}{5 : \Diamond A \supset \square\Diamond A}}
$$

Classical Rules: Example

$$
\dfrac{\mathsf{b}^{\Diamond}\,\dfrac{\Box\bar{A}, \Diamond A, [\Diamond A]}{\Box\bar{A}, [\Diamond\Diamond A, \Diamond A]} \qquad \Box\bar{A}, [\Box\Box\bar{A}, \Diamond A]}{\dfrac{\Box\,\dfrac{\Box\bar{A}, [\Diamond A]}{\Box\bar{A}, \Box\Diamond A}}{5 : \Diamond A \supset \Box\Diamond A}\,\vee}\,\mathsf{cut}
$$

Classical Rules: Example

$$
\dfrac{
  \mathsf{b}^{\Diamond}\,\dfrac{\Box\bar{A}, \Diamond A, [\Diamond A]}{\Box\bar{A}, [\Diamond\Diamond A, \Diamond A]}
  \qquad
  \Box\,\dfrac{\Box\bar{A}, [[\Box\bar{A}], \Diamond A]}{\Box\bar{A}, [\Box\Box\bar{A}, \Diamond A]}
}{
  \Box\,\dfrac{\Box\bar{A}, [\Diamond A]}{\dfrac{\Box\bar{A}, \Box\Diamond A}{5 : \Diamond A \supset \Box\Diamond A}\,\vee}
}\,\mathsf{cut}
$$

Classical Rules: Example

$$
\vee\,\dfrac{\Box\,\dfrac{\mathrm{cut}\,\dfrac{\mathrm{b}^{\Diamond}\,\dfrac{\Box\bar{A}, \Diamond A, [\Diamond A]}{\Box\bar{A}, [\Diamond\Diamond A, \Diamond A]} \qquad \Box\,\dfrac{4^{\Diamond}\,\dfrac{\Box\bar{A}, [[\Box\bar{A}, \Diamond A]]}{\Box\bar{A}, [[\Box\bar{A}], \Diamond A]}}{\Box\bar{A}, [\Box\Box\bar{A}, \Diamond A]}}{\Box\bar{A}, [\Diamond A]}}{\Box\bar{A}, \Box\Diamond A}}{5 : \Diamond A \supset \Box\Diamond A}
$$

Intuitionistic Rules

System NIK

Intuitionistic Rules

System NIK

$$\text{id}\ \frac{}{\Gamma\{a^{\bullet}, a^{\circ}\}}$$

$$\wedge^{\circ}\ \frac{\Gamma\{A^{\circ}\} \quad \Gamma\{B^{\circ}\}}{\Gamma\{A \wedge B^{\circ}\}}$$

$$\vee^{\circ}\ \frac{\Gamma\{A^{\circ}\}}{\Gamma\{A \vee B^{\circ}\}} \qquad \vee^{\circ}\ \frac{\Gamma\{B^{\circ}\}}{\Gamma\{A \vee B^{\circ}\}}$$

$$\Box^{\circ}\ \frac{\Gamma\{[A^{\circ}]\}}{\Gamma\{\Box A^{\circ}\}} \qquad \Diamond^{\circ}\ \frac{\Gamma\{[A^{\circ}, \Delta]\}}{\Gamma\{\Diamond A^{\circ}, [\Delta]\}}$$

Intuitionistic Rules

System NIK

$$\text{id}\ \frac{}{\Gamma\{a^\bullet, a^\circ\}}$$

$$\wedge^\bullet\ \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \qquad \wedge^\circ\ \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}}$$

$$\vee^\bullet\ \frac{\Gamma\{A^\bullet\} \quad \Gamma\{A^\bullet\}}{\Gamma\{A \vee B^\bullet\}} \qquad \vee^\circ\ \frac{\Gamma\{A^\circ\}}{\Gamma\{A \vee B^\circ\}} \quad \vee^\circ\ \frac{\Gamma\{B^\circ\}}{\Gamma\{A \vee B^\circ\}}$$

$$\Box^\bullet\ \frac{\Gamma\{[A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}} \quad \Diamond^\bullet\ \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}} \qquad \Box^\circ\ \frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Box A^\circ\}} \quad \Diamond^\circ\ \frac{\Gamma\{[A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}}$$

Intuitionistic Rules

System NIK

$$\bot^\bullet \frac{}{\Gamma\{\bot^\bullet\}} \qquad \mathsf{id}\ \frac{}{\Gamma\{a^\bullet, a^\circ\}}$$

$$\wedge^\bullet \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \qquad \wedge^\circ \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}}$$

$$\vee^\bullet \frac{\Gamma\{A^\bullet\} \quad \Gamma\{A^\bullet\}}{\Gamma\{A \vee B^\bullet\}} \qquad \vee^\circ \frac{\Gamma\{A^\circ\}}{\Gamma\{A \vee B^\circ\}} \quad \vee^\circ \frac{\Gamma\{B^\circ\}}{\Gamma\{A \vee B^\circ\}}$$

$$\Box^\bullet \frac{\Gamma\{[A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}} \quad \Diamond^\bullet \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}} \qquad \Box^\circ \frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Box A^\circ\}} \quad \Diamond^\circ \frac{\Gamma\{[A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}}$$

Intuitionistic Rules

System NIK

$$\bot^\bullet \ \frac{}{\Gamma\{\bot^\bullet\}} \qquad \mathsf{id} \ \frac{}{\Gamma\{a^\bullet, a^\circ\}}$$

$$\wedge^\bullet \ \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \qquad \wedge^\circ \ \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}}$$

$$\vee^\bullet \ \frac{\Gamma\{A^\bullet\} \quad \Gamma\{A^\bullet\}}{\Gamma\{A \vee B^\bullet\}} \qquad \vee^\circ \ \frac{\Gamma\{A^\circ\}}{\Gamma\{A \vee B^\circ\}} \quad \vee^\circ \ \frac{\Gamma\{B^\circ\}}{\Gamma\{A \vee B^\circ\}}$$

$$\supset^\bullet \ \frac{\Gamma^\downarrow\{A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \supset B^\bullet\}} \qquad \supset^\circ \ \frac{\Gamma\{A^\bullet, B^\circ\}}{\Gamma\{A \supset B^\circ\}}$$

$$\Box^\bullet \ \frac{\Gamma\{[A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}} \quad \Diamond^\bullet \ \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}} \qquad \Box^\circ \ \frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Box A^\circ\}} \quad \Diamond^\circ \ \frac{\Gamma\{[A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}}$$

Intuitionistic Rules

System NIK

$$\bot^\bullet \frac{}{\Gamma\{\bot^\bullet\}} \qquad \mathsf{c}\ \frac{\Gamma\{A^\bullet, A^\bullet\}}{\Gamma\{A^\bullet\}} \qquad \mathsf{id}\ \frac{}{\Gamma\{a^\bullet, a^\circ\}}$$

$$\wedge^\bullet \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \qquad \wedge^\circ \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}}$$

$$\vee^\bullet \frac{\Gamma\{A^\bullet\} \quad \Gamma\{A^\bullet\}}{\Gamma\{A \vee B^\bullet\}} \qquad \vee^\circ \frac{\Gamma\{A^\circ\}}{\Gamma\{A \vee B^\circ\}} \quad \vee^\circ \frac{\Gamma\{B^\circ\}}{\Gamma\{A \vee B^\circ\}}$$

$$\supset^\bullet \frac{\Gamma^{\downarrow}\{A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \supset B^\bullet\}} \qquad \supset^\circ \frac{\Gamma\{A^\bullet, B^\circ\}}{\Gamma\{A \supset B^\circ\}}$$

$$\Box^\bullet \frac{\Gamma\{[A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}} \quad \Diamond^\bullet \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}} \qquad \Box^\circ \frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Box A^\circ\}} \quad \Diamond^\circ \frac{\Gamma\{[A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}}$$

Intuitionistic Rules

System NIK

$$\bot^\bullet \frac{}{\Gamma\{\bot^\bullet\}} \qquad \mathsf{c}\ \frac{\Gamma\{A^\bullet, A^\bullet\}}{\Gamma\{A^\bullet\}} \qquad \mathsf{id}\ \frac{}{\Gamma\{a^\bullet, a^\circ\}}$$

$$\wedge^\bullet \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \qquad \wedge^\circ \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}}$$

$$\vee^\bullet \frac{\Gamma\{A^\bullet\} \quad \Gamma\{A^\bullet\}}{\Gamma\{A \vee B^\bullet\}} \qquad \vee^\circ \frac{\Gamma\{A^\circ\}}{\Gamma\{A \vee B^\circ\}} \qquad \vee^\circ \frac{\Gamma\{B^\circ\}}{\Gamma\{A \vee B^\circ\}}$$

$$\supset^\bullet \frac{\Gamma^\downarrow\{A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \supset B^\bullet\}} \qquad \supset^\circ \frac{\Gamma\{A^\bullet, B^\circ\}}{\Gamma\{A \supset B^\circ\}}$$

$$\Box^\bullet \frac{\Gamma\{[A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}} \qquad \Diamond^\bullet \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}} \qquad \Box^\circ \frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Box A^\circ\}} \qquad \Diamond^\circ \frac{\Gamma\{[A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}}$$

Additional structural rules

$$\mathsf{w}\ \frac{\Gamma\{\emptyset\}}{\Gamma\{\Lambda^\bullet\}} \qquad \mathsf{cut}\ \frac{\Gamma\{A^\bullet\} \quad \Gamma^\downarrow\{A^\circ\}}{\Gamma\{\emptyset\}}$$

Intuitionistic Rules

Modal $\Diamond^\circ$-rules	Modal $\Box^\bullet$-rules	Modal structural rules
$\mathsf{d}^\circ \ \dfrac{\Gamma\{[A^\circ]\}}{\Gamma\{\Diamond A^\circ\}}$	$\mathsf{d}^\bullet \ \dfrac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Box A^\bullet\}}$	$\mathsf{d}^{[]} \ \dfrac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}}$
$\mathsf{t}^\circ \ \dfrac{\Gamma\{A^\circ\}}{\Gamma\{\Diamond A^\circ\}}$	$\mathsf{t}^\bullet \ \dfrac{\Gamma\{A^\bullet\}}{\Gamma\{\Box A^\bullet\}}$	$\mathsf{t}^{[]} \ \dfrac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}}$
$\mathsf{b}^\circ \ \dfrac{\Gamma\{[\Delta], A^\circ\}}{\Gamma\{[\Delta, \Diamond A^\circ]\}}$	$\mathsf{b}^\bullet \ \dfrac{\Gamma\{[\Delta], A^\bullet\}}{\Gamma\{[\Delta, \Box A^\bullet]\}}$	$\mathsf{b}^{[]} \ \dfrac{\Gamma\{[\Sigma, [\Delta]]\}}{\Gamma\{[\Sigma], \Delta\}}$
$4^\circ \ \dfrac{\Gamma\{[\Diamond A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}}$	$4^\bullet \ \dfrac{\Gamma\{[\Box A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}}$	$4^{[]} \ \dfrac{\Gamma\{[\Delta], [\Sigma]\}}{\Gamma\{[[\Delta], \Sigma]\}}$
$5^\circ \ \dfrac{\Gamma\{\emptyset\}\{\Diamond A^\circ\}}{\Gamma\{\Diamond A^\circ\}\{\emptyset\}}$	$5^\bullet \ \dfrac{\Gamma\{\emptyset\}\{\Box A^\bullet\}}{\Gamma\{\Box A^\bullet\}\{\emptyset\}}$	$5^{[]} \ \dfrac{\Gamma\{[\Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Delta]\}}$

45-Closure

Not all combination X of the $\mathrm{d}, \mathrm{t}, \mathrm{b}, 4, 5$ rules can lead to a complete cut-free system $\mathrm{NK} \cup \mathrm{X}^{\Diamond}$ or $\mathrm{NIK} \cup \mathrm{X}^{\circ} \cup \mathrm{X}^{\bullet}$.
ex: $\{\mathrm{b}, 5\} \vdash 4 : \Diamond\Diamond A \supset \Diamond A$
but 4 is not derivable in $\mathrm{NK} \cup \{\mathrm{b}^{\Diamond}, 5^{\Diamond}\} \setminus \{\mathrm{cut}\}$

If $\mathrm{X} \subseteq \{\mathrm{d}, \mathrm{t}, \mathrm{b}, 4, 5\}$, the 45-closure is defined as:

$$\hat{\mathrm{X}} = \begin{cases} \mathrm{X} \cup \{4\} & \text{if } \{\mathrm{b}, 5\} \subseteq \mathrm{X} \text{ or if } \{\mathrm{t}, 5\} \subseteq \mathrm{X} \\ \mathrm{X} \cup \{5\} & \text{if } \{\mathrm{b}, 4\} \subseteq \mathrm{X} \\ \mathrm{X} & \text{otherwise} \end{cases}$$

Cut Elimination

Theorem: Cut-Elimination in the 45-closure

Let $X \subseteq \{d, t, b, 4, 5\}$.

- (Brünnler, 2009) If Γ is derivable in $NK \cup X^{\Diamond} \cup \{cut\}$ then it is derivable in $NK \cup \hat{X}^{\Diamond}$.
- (Straßburger, 2013) If Γ is derivable in $NIK \cup X^{\bullet} \cup X^{\circ} \cup \{cut\}$ then it is derivable in

$$\begin{cases} NIK \cup \hat{X}^{\bullet} \cup \hat{X}^{\circ} & \text{if } d \notin X \\ NIK \cup \hat{X}^{\bullet} \cup \hat{X}^{\circ} \cup \{d^{[]}\} & \text{if } d \in X \end{cases}$$

Modularity

Theorem: Modular Cut-Elimination

Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$.

- If Γ is derivable in $\mathsf{NK} \cup \mathsf{X}^{\Diamond} \cup \{\mathsf{cut}\}$ then it is derivable in $\mathsf{NK} \cup \mathsf{X}^{\Diamond} \cup \mathsf{X}^{[]}$.
- If Γ is derivable in $\mathsf{NIK} \cup \mathsf{X}^{\bullet} \cup \mathsf{X}^{\circ} \cup \{\mathsf{cut}\}$ then it is derivable in $\mathsf{NIK} \cup \mathsf{X}^{\bullet} \cup \mathsf{X}^{\circ} \cup \mathsf{X}^{[]}$.

Modularity

If Γ is derivable in $\mathsf{NK} \cup \mathsf{X} \cup \{\mathsf{cut}\}$, then we have a proof of Γ in $\mathsf{NK} \cup \hat{\mathsf{X}}^{\Diamond}$.
If $\hat{\mathsf{X}} = \mathsf{X}$, then a proof in $\mathsf{NK} \cup \hat{\mathsf{X}}^{\Diamond}$ is trivially a proof in $\mathsf{NK} \cup \mathsf{X}^{\Diamond} \cup \mathsf{X}^{[]}$.
Otherwise, we must have one of the following three cases:

- If $\{\mathsf{t}, 5\} \subseteq \mathsf{X}$ then $\hat{\mathsf{X}} = \mathsf{X} \cup \{4\}$ ...
- If $\{\mathsf{b}, 5\} \subseteq \mathsf{X}$ then $\hat{\mathsf{X}} = \mathsf{X} \cup \{4\}$ ...
- If $\{\mathsf{b}, 4\} \subseteq \mathsf{X}$ then $\hat{\mathsf{X}} = \mathsf{X} \cup \{5\}$ ...

Modularity

- If $\{t, 5\} \subseteq X$ then $\hat{X} = X \cup \{4\}$ and the $4^\Diamond$-rule is admissible in $NK \cup X^\Diamond$.

$$4^\Diamond \frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \quad \rightsquigarrow \quad \mathsf{t}^{[]} \frac{5^\Diamond \dfrac{\mathsf{w} \dfrac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{[\emptyset], [\Diamond A, \Delta]\}}}{\Gamma\{[\Diamond A], [\Delta]\}}}{\Gamma\{\Diamond A, [\Delta]\}}$$

- If $\{b, 5\} \subseteq X$ then $\hat{X} = X \cup \{4\}$ and the $4^\Diamond$-rule is admissible in $NK \cup X^\Diamond$.

$$4^\Diamond \frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \quad \rightsquigarrow \quad \mathsf{b}^{[]} \frac{5^\Diamond \dfrac{\mathsf{w} \dfrac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{[[\emptyset], \Diamond A, \Delta]\}}}{\Gamma\{[[\Diamond A], \Delta]\}}}{\Gamma\{\Diamond A, [\Delta]\}}$$

Modularity

- If $\{b, 4\} \subseteq X$ then $\hat{X} = X \cup \{5\}$. We replace the $5^\Diamond$ by the equivalent set of rules $\{5_1^\Diamond, 5_2^\Diamond, 5_3^\Diamond\}$ and show that all three are admissible in $NK \cup X^\Diamond$.

$$5_1^\Diamond \ \frac{\Gamma\{[\Delta], \Diamond A\}}{\Gamma\{[\Delta, \Diamond A]\}} \qquad 5_2^\Diamond \ \frac{\Gamma\{[\Delta], [\Diamond A, \Sigma]\}}{\Gamma\{[\Delta, \Diamond A], [\Sigma]\}} \qquad 5_3^\Diamond \ \frac{\Gamma\{[\Delta, [\Diamond A, \Sigma]]\}}{\Gamma\{[\Delta, \Diamond A, [\Sigma]]\}}$$

Concluding Remarks

- We used both logical and structural rules to get a modular cut-free system but for some combinations of axioms only the structural or the logical rules would be sufficient depending on the system.
- In order to better understand this phenomenon, we need to find a general pattern for translating axioms into rules and to investigate for which type of axioms such a translation is possible.