Tidy proof systems for intuitionistic modal logic

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Inria, LIX, École Polytechnique

June 16, 2016

Modular focused proof systems for intuitionistic modal logic

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The quest

Modal logics:
program verification, artificial intelligence, distributed systems ...

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We want to study *automated* proof search for modal logics with a *proof-theoretically* justified approach.

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We want to study *automated* proof search for modal logics with a *proof-theoretically* justified approach.

Our specific desiderata:

1. structural proof systems (sequent style)
2. analytic (cut-free)
3. modular for a large class of modal logics
4. control of non-deterministic choices

The class of modal logics

Formulas: $A ::= a \mid A \wedge A \mid \top \mid A \vee A \mid \bot \mid A \rightarrow A$

Logic IK**:** Intuitionistic Propositional Logic

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Logic IK**:** Intuitionistic Propositional Logic + Axioms

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Kripke semantics: (Bi)relational structures

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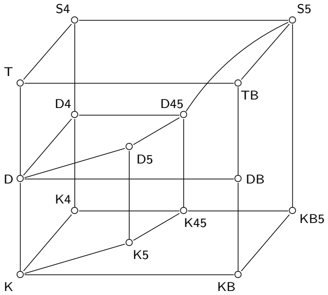
Logic IK**:** Intuitionistic Propositional Logic + Axioms

Kripke semantics: (Bi)relational structures

S5-**cube:**

d:	$\Box A \to \Diamond A$	(Seriality)
t:	$(A \to \Diamond A) \wedge \Box A \to A)$	(Reflexivity)
b:	$(A \to \Box\Diamond A) \wedge (\Diamond\Box A \to A)$	(Symmetry)
4:	$(\Diamond\Diamond A \to \Diamond A) \wedge (\Box A \to \Box\Box A)$	(Transitivity)
5:	$(\Diamond A \to \Box\Diamond A) \wedge (\Diamond\Box A \to \Box A)$	(Euclideanness)



Nested sequents

Sequent:

$$A, B, C$$

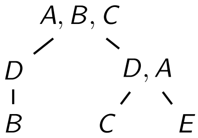
Nested sequents

Nested sequent:

$$
\begin{tikzcd}
& A, B, C \arrow[dl, dash] \arrow[dr, dash] & & \\
D \arrow[d, dash] & & D, A \arrow[dl, dash] \arrow[dr, dash] & \\
B & C & & E
\end{tikzcd}
$$

Nested sequents

Nested sequent:



$$\Gamma = A, B, C, [D, [B]], [D, A, [C], [E]]$$

Nested sequents

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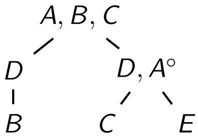
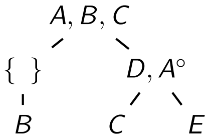


$$\Gamma = A, B, C, [D, [B]], [D, A^\circ, [C], [E]]$$

Sequent context:



$$\Gamma\{\ \} = A, B, C, [\{\ \}, [B]], [D, A^\circ, [C], [E]]$$

Nested sequents

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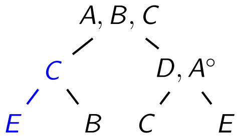
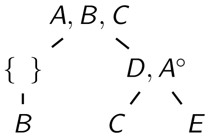


$$\Gamma\{C, [E]\} = A, B, C, [C, [E], [B]], [D, A^\circ, [C], [E]]$$

Nested sequents

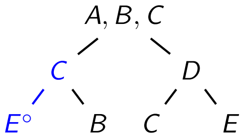
Sequent context:



$$\Gamma\{\ \} = A, B, C, [\{\ \}, [B]], [D, A^\circ, [C], [E]]$$

Nested sequents

Sequent context:



$$\Gamma^*\{C, [E^\circ]\} = A, B, C, [C, [E^\circ], [B]], [D, [C], [E]]$$

The standard nested system

System NIK:

$$\mathsf{id}\ \frac{}{\Gamma\{a, a^\circ\}} \qquad \wedge_R\ \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}} \qquad \wedge_L\ \frac{\Gamma\{A, B\}}{\Gamma\{A \wedge B\}} \qquad \top_R\ \frac{}{\Gamma\{\top^\circ\}}$$

$$\vee_{R1}\ \frac{\Gamma\{A^\circ\}}{\Gamma\{A \vee B^\circ\}} \qquad \vee_{R2}\ \frac{\Gamma\{B^\circ\}}{\Gamma\{A \vee B^\circ\}} \qquad \vee_L\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \vee B\}} \qquad \bot_L\ \frac{}{\Gamma\{\bot\}}$$

$$\to_R\ \frac{\Gamma\{A, B^\circ\}}{\Gamma\{A \to B^\circ\}} \qquad \to_L\ \frac{\Gamma^*\{A \to B, A^\circ\} \quad \Gamma\{B\}}{\Gamma\{A \to B\}}$$

$$\Diamond_R\ \frac{\Gamma\{[A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}} \qquad \Diamond_L\ \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \Box_R\ \frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Box A^\circ\}} \qquad \Box_L\ \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Box A, [\Delta]\}}$$

The standard nested system

Sequent-like rules:

$$\wedge_R \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \quad \rightsquigarrow \quad \wedge_R \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}}$$

$$\wedge_L \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C} \quad \rightsquigarrow \quad \wedge_L \frac{\Gamma\{A, B\}}{\Gamma\{A \wedge B\}}$$

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Nested rules:

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Analyticity: the cut-rule $\dfrac{\Gamma\{A\} \quad \Gamma^*\{A^\circ\}}{\Gamma\{\emptyset\}}$ is admissible

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Soundness: each rule preserves the corresponding formula

Completeness: each modal theorem has a proof in NIK

Analyticity: the cut-rule $\dfrac{\Gamma\{A\} \quad \Gamma^*\{A^\circ\}}{\Gamma\{\emptyset\}}$ is admissible

Modularity: each axiom becomes a rule

The standard nested system

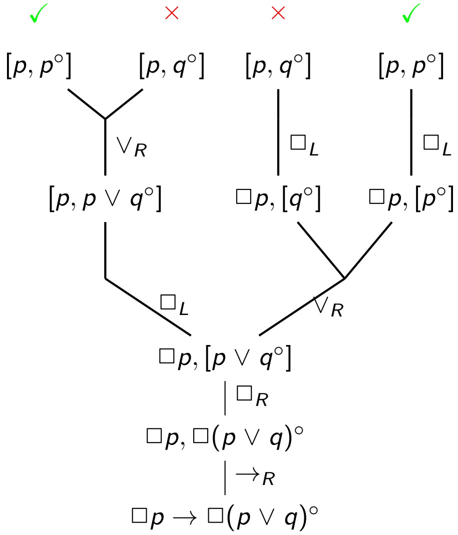
Modal rules:

$$\mathsf{d}_R\ \frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Diamond A^\circ\}} \qquad \mathsf{t}_R\ \frac{\Gamma\{A^\circ\}}{\Gamma\{\Diamond A^\circ\}} \qquad \mathsf{b}_R\ \frac{\Gamma\{[\Delta], A^\circ\}}{\Gamma\{[\Delta, \Diamond A^\circ]\}} \qquad 4_R\ \frac{\Gamma\{[\Diamond A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}} \qquad 5_R\ \frac{\Gamma\{\emptyset\}\{\Diamond A^\circ\}}{\Gamma\{\Diamond A^\circ\}\{\emptyset\}}$$

$$\mathsf{d}_L\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \mathsf{t}_L\ \frac{\Gamma\{A\}}{\Gamma\{\Box A\}} \qquad \mathsf{b}_L\ \frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Box A]\}} \qquad 4_L\ \frac{\Gamma\{[\Box A, \Delta]\}}{\Gamma\{\Box A, [\Delta]\}} \qquad 5_L\ \frac{\Gamma\{\emptyset\}\{\Box A\}}{\Gamma\{\Box A\}\{\emptyset\}}$$

d: $\Box A \to \Diamond A$	t: $A \to \Diamond A$	b: $A \to \Box\Diamond A$	4: $\Diamond\Diamond A \to \Diamond A$	5: $\Diamond A \to \Box\Diamond A$
	$\wedge\ \Box A \to A$	$\wedge\ \Diamond\Box A \to A$	$\wedge\ \Box A \to \Box\Box A$	$\wedge\ \Diamond\Box A \to \Box A$

Non-determinism and proof search space



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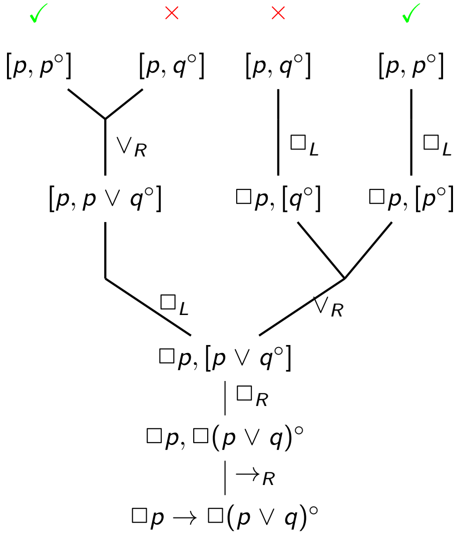

$$\to_R \; \frac{\Gamma\{A, B^\circ\}}{\Gamma\{A \to B^\circ\}}$$

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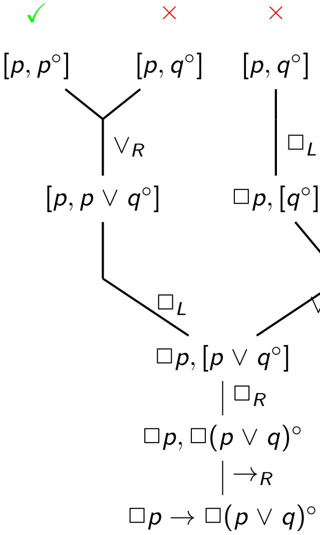

$$\Box_R \; \frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Box A^\circ\}}$$

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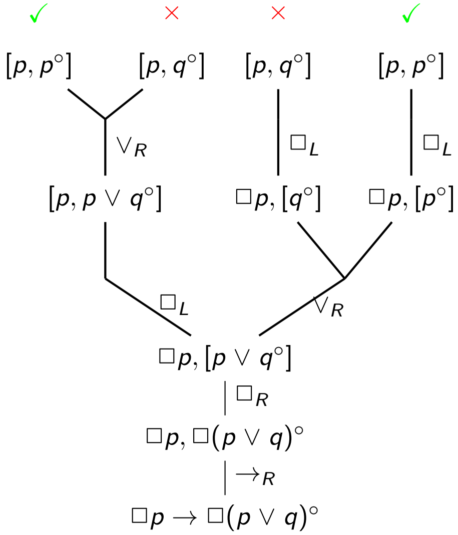

$$\Box_L \ \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Box A, [\Delta]\}}$$

$$\Box_R \ \frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Box A^\circ\}}$$

$$\to_R \ \frac{\Gamma\{A, B^\circ\}}{\Gamma\{A \to B^\circ\}}$$

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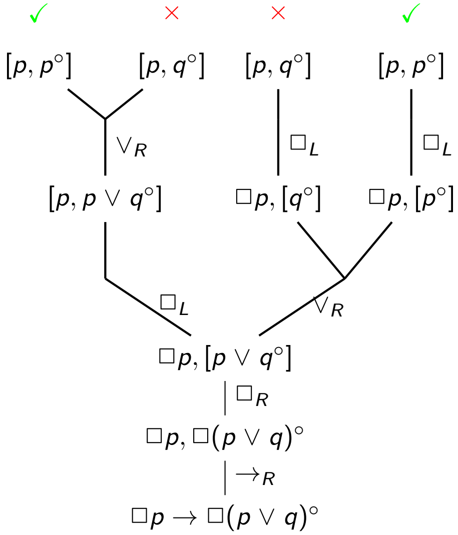

$$\vee_{R1}\ \frac{\Gamma\{A^\circ\}}{\Gamma\{A \vee B^\circ\}}$$

$$\vee_{R2}\ \frac{\Gamma\{B^\circ\}}{\Gamma\{A \vee B^\circ\}}$$

$$\Box_L\ \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Box A, [\Delta]\}}$$

$$\Box_R\ \frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Box A^\circ\}}$$

$$\rightarrow_R\ \frac{\Gamma\{A, B^\circ\}}{\Gamma\{A \rightarrow B^\circ\}}$$

Non-determinism and connectives

Polarities:			
	non-invertible right rules	:	positive connectives
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Polarized formulas:

$$\begin{array}{rcl} P, Q & ::= & p \mid Q \wedge P \mid \top \mid P \vee Q \mid \bot \\ N & ::= & n \mid P \to N \end{array}$$

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N & ::= & n \mid P \to N \mid \Box N \mid {\uparrow} P
\end{array}
$$

Focused nested sequents

Two kind of sequents:

$\Gamma\{A\}$	ordinary		
$\Gamma\{\langle N \rangle\}$	left-focused	$\Gamma\{\langle P \rangle\}$	right-focused

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$$\mathsf{id}\ \frac{}{\Gamma\{a, a^\circ\}} \qquad \to_R\ \frac{\Gamma\{A, B^\circ\}}{\Gamma\{A \to B^\circ\}} \qquad \to_L\ \frac{\Gamma^*\{A \to B, A^\circ\} \quad \Gamma\{B\}}{\Gamma\{A \to B\}}$$

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Focused system FoNIK:

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$$\mathrm{id}_R\ \frac{}{\Gamma\{p, \langle p\rangle\}} \qquad \mathrm{id}_L\ \frac{}{\Gamma\{n, \langle n\rangle\}} \qquad \to_R\ \frac{\Gamma\{P, N\}}{\Gamma\{P \to N\}} \qquad \to_L\ \frac{\Gamma\{\langle P\rangle\} \quad \Gamma\{\langle N\rangle\}}{\Gamma\{\langle P \to N\rangle\}}$$

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$$\wedge_R \frac{\Gamma\{\langle P\rangle\} \quad \Gamma\{\langle Q\rangle\}}{\Gamma\{\langle P \wedge Q\rangle\}} \qquad \wedge_L \frac{\Gamma\{P, Q\}}{\Gamma\{P \wedge Q\}} \qquad \top_R \frac{}{\Gamma\{\langle \top\rangle\}} \qquad \top_L \frac{\Gamma\{\emptyset\}}{\Gamma\{\top\}}$$

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Focused nested sequents

Modal rules:

$$\mathsf{d_R}\ \frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Diamond A^\circ\}} \qquad \mathsf{t_R}\ \frac{\Gamma\{A^\circ\}}{\Gamma\{\Diamond A^\circ\}} \qquad \mathsf{b_R}\ \frac{\Gamma\{[\Delta], A^\circ\}}{\Gamma\{[\Delta, \Diamond A^\circ]\}} \qquad \mathsf{4_R}\ \frac{\Gamma\{[\Diamond A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}} \qquad \mathsf{5_R}\ \frac{\Gamma\{\emptyset\}\{\Diamond A^\circ\}}{\Gamma\{\Diamond A^\circ\}\{\emptyset\}}$$

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$$\mathsf{d_L}\ \frac{\Gamma\{[\langle N\rangle]\}}{\Gamma\{\langle\Box N\rangle\}} \qquad \mathsf{t_L}\ \frac{\Gamma\{\langle N\rangle\}}{\Gamma\{\langle\Box N\rangle\}} \qquad \mathsf{b_L}\ \frac{\Gamma\{[\Delta], \langle N\rangle\}}{\Gamma\{[\Delta, \langle\Box N\rangle]\}} \qquad \mathsf{4_L}\ \frac{\Gamma\{[\langle\Box N\rangle, \Delta]\}}{\Gamma\{\langle\Box N\rangle, [\Delta]\}} \qquad \mathsf{5_L}\ \frac{\Gamma\{\emptyset\}\{\langle\Box N\rangle\}}{\Gamma\{\langle\Box N\rangle\}\{\emptyset\}}$$

Focused nested sequents

Depolarized sequent $\lfloor \Gamma \rfloor$: erase $\langle\,\rangle$, $\uparrow$, $\downarrow$

Soundness and completeness: NIK proves $\lfloor \Gamma \rfloor$ iff FoNIK proves Γ

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Proof of completeness: every NIK rule is admissible in FoNIK

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SyNIK: conclusion

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1. structural proof systems (sequent style)
2. analytic (cut-free)
3. modular for a large class of modal logics
4. control of non-deterministic choices

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Tidy:

$$\dfrac{\dfrac{\dfrac{}{\Gamma\{\langle\Delta\rangle\}}\,\Big\Vert\,\text{only structure}}{\Gamma\{\langle P\rangle\}}\,\Big\Vert\,\text{only logic}}{\Gamma\{\uparrow P\}}$$