Comparing $\Box$ and ! via polarities

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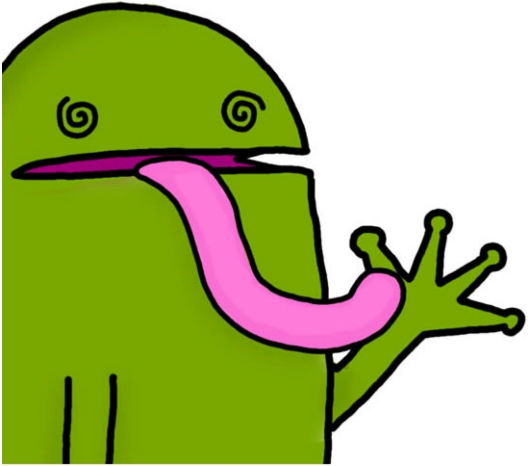
The answer

The answer

"from a proof-theoretical point of view
exponentials behave exactly like S4 modalities"

[Martini & Masini, 1994]

Wait...what? woo hoo.



Are ! and $\Box$ interchangeable?

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$A ::= x \mid x^\perp \mid A \wedge A \mid \top \mid A \vee A \mid \bot$

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Theorem: [Martini & Masini, 1994]

$$\Gamma \text{ provable in S4} \quad \Leftrightarrow \quad \Gamma^+ \text{ provable in LL}$$

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Their answer:

cut-free proof of an $S4$ sequent

$$\Updownarrow$$

cut-free proof of its LL translation

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Our question:

focused polarised
cut-free proof of an $S4$ sequent

$\Updownarrow$

focused polarised
cut-free proof of its LL translation

?

Polarity and focusing

Polarities:	non-invertible rules	:	positive connectives
	invertible rules	:	negative connectives

[Andreoli, 1990] [Laurent, 2004]

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Inversion: in $\begin{array}{c}\pi \| \\ \vdash N, \Gamma\end{array}$ the last rule is negative.

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Focus on a positive formula:

in $\dfrac{\pi \Vert}{\vdash P, \Gamma}$ only rules decomposing P between two rules decomposing P

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Completeness of focusing:

if a formula F is provable then F has a focused proof

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[Miller, Volpe, 2015] [Chaudhuri, M., Strassburger, 2016]

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This is...

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...not the same!

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Modular focused systems for modal logics

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Classical normal modal logics:

k: $\Box(A \to B) \to (\Box A \to \Box B)$

d: $\Box A \to \Diamond A$ (Seriality)
t: $\Box A \to A$ (Reflexivity)
b: $\Diamond\Box A \to A$ (Symmetry)
4: $\Box A \to \Box\Box A$ (Transitivity)
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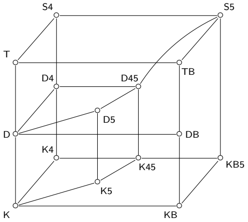

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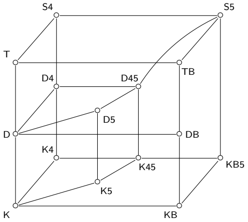

Nested sequent system:

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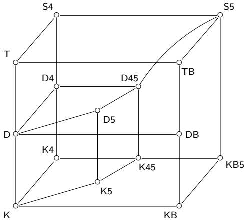

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1. complete and modular
   F is a theorem of K + axioms iff F is provable in KN + rules

[Brünnler, 2009]

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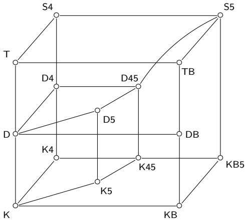


Nested sequent system:

1. complete and modular
   F is a theorem of K + axioms iff F is provable in KN + rules
   [Brünnler, 2009]
2. polarised and focused
   F theorem of K + axioms iff F has a focused proof in KN + rules
   [Chaudhuri, M., Strassburger, 2016]

Nested sequents

Nested sequents generalise sequents from a multiset of formulas

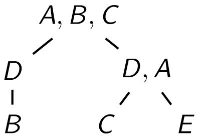
Sequent:

$$A, B, C$$

Nested sequents

Nested sequents generalise sequents from a multiset of formulas to a tree of multisets of formulas.

Nested sequent:



Nested sequents

In the sequent term, brackets indicate the parent-child relation in the tree

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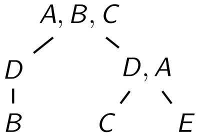

$$\Gamma = A, B, C, [D, [B]], [D, A, [C], [E]]$$

Nested sequents

In the sequent term, brackets indicate the parent-child relation in the tree and can be interpreted as the modal $\Box$.

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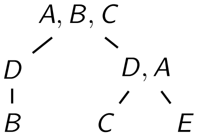


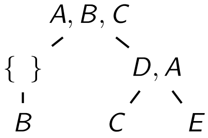
$$\Gamma = A, B, C, [D, [B]], [D, A, [C], [E]]$$

$$A \lor B \lor C \lor \Box(D \lor \Box B), \Box(D \lor A \lor \Box C \lor \Box E)$$

Nested sequents

A context is obtained by removing a formula and replacing it by a hole

Sequent context:



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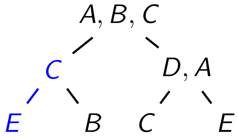
Nested sequents

A context is obtained by removing a formula and replacing it by a hole that can then be filled by another nested sequent.

Sequent context:

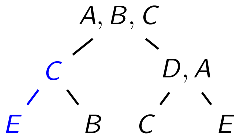


$$\Gamma\{C, [E]\} = A, B, C, [C, [E], [B]], [D, A, [C], [E]]$$

Nested sequents

This allows us to build rules than can be applied at any depth in the tree.

Sequent context:



$$\Gamma\{C, [E]\} = A, B, C, [C, [E], [B]], [D, A, [C], [E]]$$

The standard nested system for modal logics

Formulas: $A \quad ::= \quad x \mid x^{\perp} \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A$

System KN**:**

$$\mathsf{c}\ \frac{\Gamma\{A, A\}}{\Gamma\{A\}} \qquad \Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

$$\mathsf{id}\ \frac{}{\Gamma\{x^{\perp}, x\}} \qquad \Diamond_{\mathsf{k}}\ \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}$$

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Modal rules:

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[Brünnler, 2009]

The focused nested system for modal logics

Polarized formulas:

$$P ::= x \mid A \stackrel{+}{\wedge} A \mid A \stackrel{+}{\vee} A \mid \Diamond A$$
$$N ::= x^{\perp} \mid A \stackrel{-}{\vee} A \mid A \stackrel{-}{\wedge} A \mid \Box A$$

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$$\Box \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \overset{-}{\vee} \frac{\Gamma\{A, B\}}{\Gamma\{A \overset{-}{\vee} B\}} \qquad \overset{-}{\wedge} \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \overset{-}{\wedge} B\}}$$

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Focused system KNF**:**

$$\Box\,\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \overset{-}{\vee}\,\frac{\Gamma\{A, B\}}{\Gamma\{A \overset{-}{\vee} B\}} \qquad \overset{-}{\wedge}\,\frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \overset{-}{\wedge} B\}} \qquad \mathsf{dec}\,\frac{\Gamma\{P, \langle P\rangle\}}{\Gamma\{P\}}$$

$$\mathsf{id}\,\frac{}{\Gamma\{x^{\perp}, \langle x\rangle\}} \qquad \Diamond_k\,\frac{\Gamma\{[\langle A\rangle, \Delta]\}}{\Gamma\{\langle \Diamond A\rangle, [\Delta]\}} \qquad \wedge\,\frac{\Gamma\{\langle A\rangle\} \quad \Gamma\{\langle B\rangle\}}{\Gamma\{\langle A \overset{+}{\wedge} B\rangle\}} \qquad \overset{+}{\vee}_i\,\frac{\Gamma\{\langle A_i\rangle\}}{\Gamma\{\langle A_1 \overset{+}{\vee} A_2\rangle\}} \qquad \mathsf{rel}\,\frac{\Gamma\{N\}}{\Gamma\{\langle N\rangle\}}$$

Modal rules:

$$\Diamond_{\mathsf{d}}\,\frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \Diamond_{\mathsf{t}}\,\frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}} \qquad \Diamond_{\mathsf{b}}\,\frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Diamond A]\}} \qquad \Diamond_{4}\,\frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad \Diamond_{5}\,\frac{\Gamma\{\emptyset\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}$$

The focused nested system for modal logics

Polarized formulas:

$$P ::= x \mid A \overset{+}{\wedge} A \mid A \overset{+}{\vee} A \mid \Diamond A$$
$$N ::= x^\perp \mid A \bar{\vee} A \mid A \bar{\wedge} A \mid \Box A$$

Focused system KNF**:**

$$\Box \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \bar{\vee} \frac{\Gamma\{A, B\}}{\Gamma\{A \bar{\vee} B\}} \qquad \bar{\wedge} \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \bar{\wedge} B\}} \qquad \mathsf{dec} \frac{\Gamma\{P, \langle P \rangle\}}{\Gamma\{P\}}$$

$$\mathsf{id} \frac{}{\Gamma\{x^\perp, \langle x \rangle\}} \qquad \Diamond_k \frac{\Gamma\{[\langle A \rangle, \Delta]\}}{\Gamma\{\langle \Diamond A \rangle, [\Delta]\}} \qquad \overset{+}{\wedge} \frac{\Gamma\{\langle A \rangle\} \quad \Gamma\{\langle B \rangle\}}{\Gamma\{\langle A \overset{+}{\wedge} B \rangle\}} \qquad \overset{+}{\vee}_i \frac{\Gamma\{\langle A_i \rangle\}}{\Gamma\{\langle A_1 \overset{+}{\vee} A_2 \rangle\}} \qquad \mathsf{rel} \frac{\Gamma\{N\}}{\Gamma\{\langle N \rangle\}}$$

Focused modal rules:

$$\Diamond_d \frac{\Gamma\{[\langle A \rangle]\}}{\Gamma\{\langle \Diamond A \rangle\}} \qquad \Diamond_t \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \Diamond A \rangle\}} \qquad \Diamond_b \frac{\Gamma\{[\Delta], \langle A \rangle\}}{\Gamma\{[\Delta, \langle \Diamond A \rangle]\}} \qquad \Diamond_4 \frac{\Gamma\{[\langle \Diamond A \rangle, \Delta]\}}{\Gamma\{\langle \Diamond A \rangle, [\Delta]\}} \qquad \Diamond_5 \frac{\Gamma\{\emptyset\}\{\langle \Diamond A \rangle\}}{\Gamma\{\langle \Diamond A \rangle\}\{\emptyset\}}$$

A nested system for MELL

Formulas: $A ::= x \mid x^\perp \mid A \otimes A \mid 1 \mid A \parr A \mid \perp \mid !A \mid ?A$

System NMELL:

$$\mathsf{id}\ \frac{}{\Gamma^{[\,]}\{x, x^\perp\}} \qquad 1\ \frac{}{\Gamma^{[\,]}\{1\}}$$

$$\perp\ \frac{\Gamma\{\emptyset\}}{\Gamma\{\perp\}} \qquad \parr\ \frac{\Gamma\{A, B\}}{\Gamma\{A \parr B\}} \qquad \otimes\ \frac{\Gamma\{A\} \quad \Delta\{B\}}{\Gamma \cdot \Delta\{A \otimes B\}} \qquad !\ \frac{\Gamma\{[A]\}}{\Gamma\{!A\}}$$

$$?_{\mathsf{t}}\ \frac{\Gamma\{A\}}{\Gamma\{?A\}} \qquad ?_4\ \frac{\Gamma\{[?A, \Delta]\}}{\Gamma\{?A, [\Delta]\}} \qquad ?_{\mathsf{c}}\ \frac{\Gamma\{?A, ?A\}}{\Gamma\{?A\}} \qquad ?_{\mathsf{w}}\ \frac{\Gamma\{\emptyset\}}{\Gamma\{?A\}}$$

1. $\Gamma^{[\,]}\{\ \} ::= \{\ \} \mid [\Gamma^{[\,]}\{\ \}]$
2. merge $\Gamma \cdot \Delta\{\ \}$ when $\mathit{depth}(\Gamma\{\ \}) = \mathit{depth}(\Delta\{\ \})$

A nested system for MELL

Exponentials:

$$? \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} \qquad ?_t \frac{\Gamma\{A\}}{\Gamma\{?A\}}$$

A nested system for MELL

Exponentials:

$$? \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} \qquad ?_t \frac{\Gamma\{A\}}{\Gamma\{?A\}}$$

$$! \frac{\vdash ?\Delta, A}{\vdash ?\Delta, !A}$$

$$\begin{array}{c} \Gamma\{[?B_1, \ldots, ?B_n, A]\} \\ ?_4 \Vert \\ ?_4 \dfrac{\Gamma\{?B_1, \ldots, ?B_{n-1}, [?B_n, A]\}}{! \dfrac{\Gamma\{?B_1, \ldots, ?B_n, [A]\}}{\Gamma\{?B_1, \ldots, ?B_n, !A\}}} \end{array}$$

Could ! be negative like $\Box$?

Formulas: $A ::= x \mid x^\perp \mid A \otimes A \mid 1 \mid A \parr A \mid \bot \mid !A \mid ?A$

Could ! be negative like $\Box$?

Polarized formulas:

$$
\begin{array}{lcl}
P & ::= & x \mid A \otimes A \mid 1 \mid ?A \\
N & ::= & x^{\perp} \mid A \parr A \mid \perp \mid !A
\end{array}
$$

Could ! be negative like $\Box$?

Polarized formulas:

$$P \quad ::= \quad x \mid A \otimes A \mid 1 \mid ?A$$
$$N \quad ::= \quad x^{\perp} \mid A \parr A \mid \bot \mid !A$$

A critical example:

$$\dfrac{\dfrac{1\,\dfrac{}{\langle 1 \rangle} \qquad \langle ?x^{\perp} \rangle, !x \otimes !x}{\langle 1 \otimes ?x^{\perp} \rangle, !x \otimes !x}\,\otimes}{1 \otimes ?x^{\perp}, !x \otimes !x}\,\text{dec}$$

Could ! be negative like □?

Polarized formulas:

$$
\begin{array}{lcl}
P & ::= & x \mid A \otimes A \mid 1 \mid ?A \\
N & ::= & x^{\perp} \mid A \parr A \mid \perp \mid !A
\end{array}
$$

A critical example:

$$
\text{dec}\;\dfrac{\otimes\;\dfrac{1\;\dfrac{}{\langle 1 \rangle} \qquad \langle ?x^{\perp} \rangle, !x \otimes !x}{\langle 1 \otimes ?x^{\perp} \rangle, !x \otimes !x}}{1 \otimes ?x^{\perp}, !x \otimes !x}
$$

$$
\otimes\;\dfrac{\otimes\;\dfrac{1\;\dfrac{}{1} \qquad ?_c\;\dfrac{\otimes\;\dfrac{!\;\dfrac{?_4\;\dfrac{?_t\;\dfrac{\text{id}\;\dfrac{}{[x^{\perp}, x]}}{[?x^{\perp}, x]}}{?x^{\perp}, [x]}}{?x^{\perp}, !x} \qquad !\;\dfrac{?_4\;\dfrac{?_t\;\dfrac{\text{id}\;\dfrac{}{[x^{\perp}, x]}}{[?x^{\perp}, x]}}{?x^{\perp}, [x]}}{?x^{\perp}, !x}}{?x^{\perp}, ?x^{\perp}, !x \otimes !x}}{?x^{\perp}, !x \otimes !x}}{1 \otimes ?x^{\perp}, !x \otimes !x}}{1 \otimes ?x^{\perp}, !x \otimes !x}
$$

Conclusion

Exponentials do not behave like S4 modalities in terms of polarities

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Linear logic

$$\frac{}{\vdash \Gamma, a, \bar{a}} \quad \frac{}{\vdash 1} \quad \frac{}{\vdash \Gamma, \top} \quad \frac{\Gamma}{\vdash \Gamma, \bot}$$

$$\otimes\,\frac{\vdash \Gamma_1, A \quad \vdash \Gamma_2, B}{\vdash \Gamma_1, \Gamma_2, A \otimes B} \quad \parr\,\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B} \quad \&\,\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B} \quad \oplus_1\,\frac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B} \quad \oplus_2\,\frac{\vdash \Gamma, B}{\vdash \Gamma, A \vee B}$$

$$?\,\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} \quad !\,\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} \quad \mathrm{c}\,\frac{\vdash \Gamma}{\vdash \Gamma, ?A} \quad \mathrm{w}\,\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}$$

The problem with adding additives

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{[?x^{\perp}, x]}\,\text{id}}{?x^{\perp}, [x]}\,?}{?x^{\perp} \oplus ?x, [x]}\,\oplus_1 \quad \dfrac{\dfrac{\dfrac{}{[?x, x^{\perp}]}\,\text{id}}{?x, [x^{\perp}]}\,?}{?x \oplus ?x, [x^{\perp}]}\,\oplus_2}{\dfrac{?x^{\perp} \oplus ?x, [x \,\&\, x^{\perp}]}{?x^{\perp} \oplus ?x, !(x \,\&\, x^{\perp})}\,!}\,\&
$$