From axioms to synthetic inference rules via focusing

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Joint work with Dale Miller, Elaine Pimentel, and Marco Volpe

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The axioms-as-rules problem

How to incorporate inference rules encoding axioms into existing proof systems for classical and intuitionistic logics?

A fresh view to an old problem:
The combination of bipolars and focusing provides simple rules for atomic formulas.

Motivation

Object

Reasoning

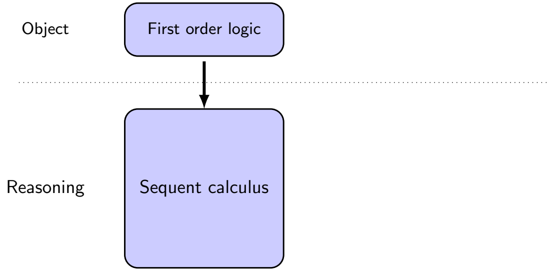
Motivation

Object

First order logic

Reasoning

Motivation



Avantages of the sequent framework

(1) simple; (2) strong properties (*analyticity*); (3) easy implementation.

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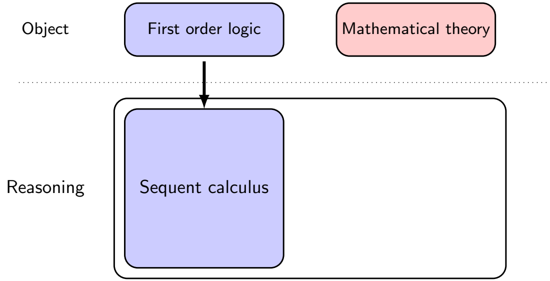

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Add mathematical theories to first order logic

Reason about them using all the machinery already built for the sequent framework.

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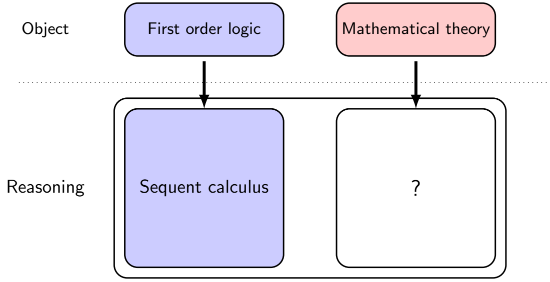

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- ⋆ Long tradition: Negri, von Plato, Dyckhoff; Simpson, Viganò, Ciabattoni; Dowek...

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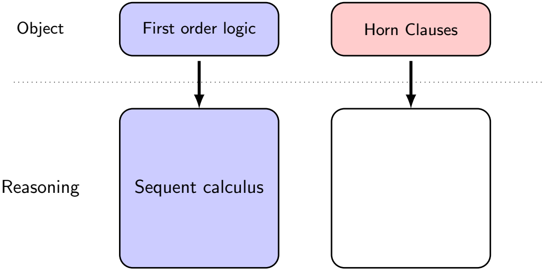

Horn Clauses: $\forall \overline{z}(P_1 \wedge \ldots \wedge P_m \rightarrow Q)$

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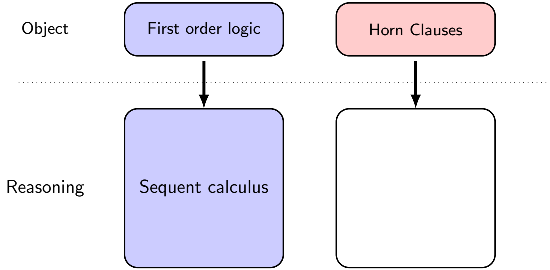

Horn Clauses: $\forall \overline{z}(P_1 \wedge \ldots \wedge P_m \rightarrow Q)$

▷ In logic programming: To show Q, one needs to show P_1 and ... and P_m.

$$\frac{\Gamma \vdash P_1 \quad \ldots \quad \Gamma \vdash P_m}{\Gamma \vdash Q}$$ (back)

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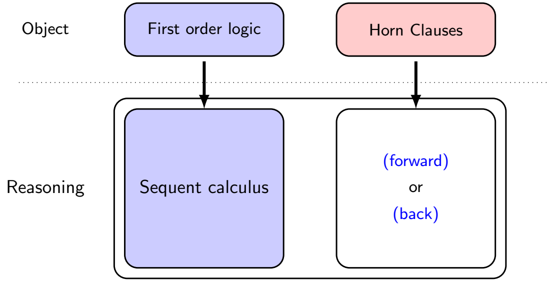

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$$\frac{\Gamma \vdash P_1 \quad \ldots \quad \Gamma \vdash P_m}{\Gamma \vdash Q} \text{ (back)}$$

- In theorem proving: To infer C from $P_1, \ldots, P_m$, it is enough to infer it from Q.

$$\frac{Q, \Gamma \vdash C}{P_1, \ldots, P_m, \Gamma \vdash C} \text{ (forward)}$$

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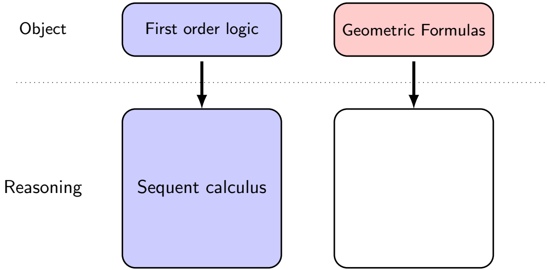

Geometric/Coherent Implications:

$$\forall\vec{y}.(\ P_1 \wedge \ldots \wedge P_m\ \rightarrow\ \exists\vec{x_1}.(Q_{11} \wedge \ldots Q_{1k_1}) \vee \ldots \vee \exists\vec{x_n}.(Q_{n1} \wedge \ldots Q_{nk_n})\ )$$

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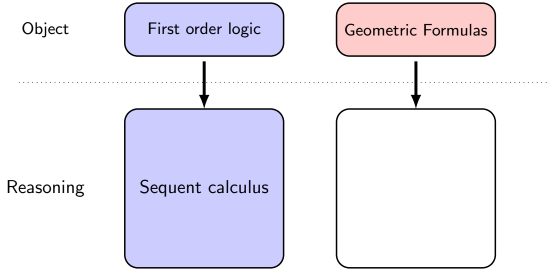

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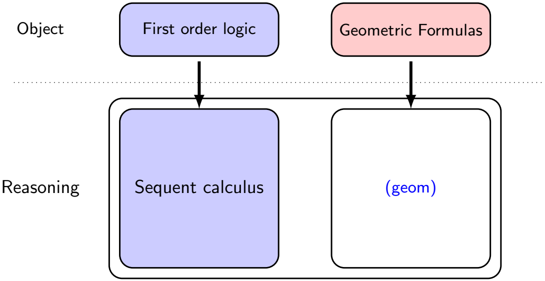

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- ▷ Originally a category theoretic notion $\rightsquigarrow$ structural proof theory [Negri-von Plato]
- ▷ "A certain simple form into which only atomic formulas play a critical part" [Simpson]

$$\frac{Q_{11}, \ldots, Q_{1k_1}, \Gamma \vdash C \quad \ldots \quad Q_{n1}, \ldots, Q_{nk_n}, \Gamma \vdash C}{P_1, \ldots, P_m, \Gamma \vdash C}\ (\text{geom})$$

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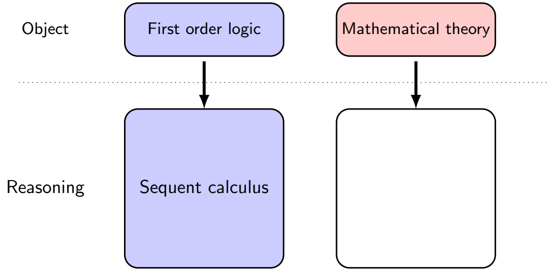
Object
First order logic
Mathematical theory
Reasoning
Sequent calculus

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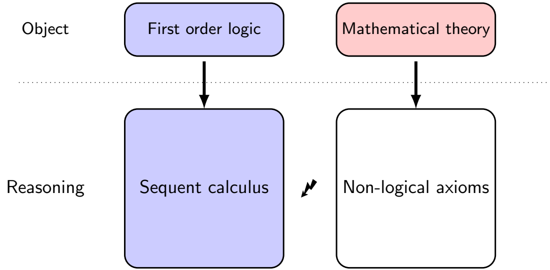

Non-logical axioms.

Take $\vdash P \to Q$ and $\vdash P$. Then, cut-elimination would fail.[Girard]

$$cut\ \dfrac{\dfrac{}{\vdash P} \qquad cut\ \dfrac{\dfrac{}{\vdash P \to Q} \qquad \to_{L}\ \dfrac{\dfrac{}{P \vdash P} \quad \dfrac{}{Q \vdash Q}}{P, P \to Q \vdash Q}}{P \vdash Q}}{\vdash Q}$$

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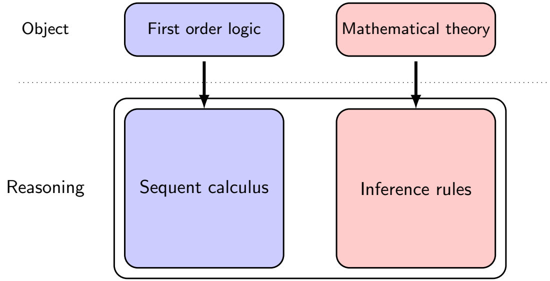


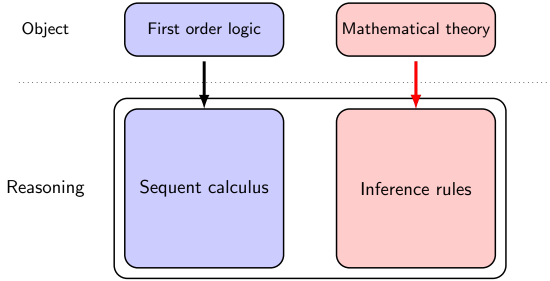
Non-logical rules of inference.

$$\frac{\Gamma, Q \vdash C}{\Gamma, P \vdash C}\ (P \rightarrow Q) \qquad \frac{\Gamma, P \vdash C}{\Gamma \vdash C}\ (P)$$

Now, Q has a direct cut-free proof

$$\cfrac{\cfrac{\cfrac{}{Q \vdash Q}}{P \vdash Q}\ (P \rightarrow Q)}{\vdash Q}\ (P)$$

In this talk



Which ones, how, and why?

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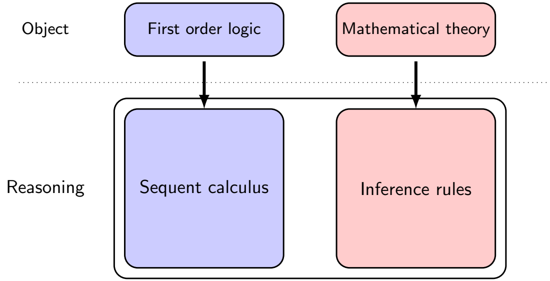

bipolar axioms + focusing = synthetic rules for atoms

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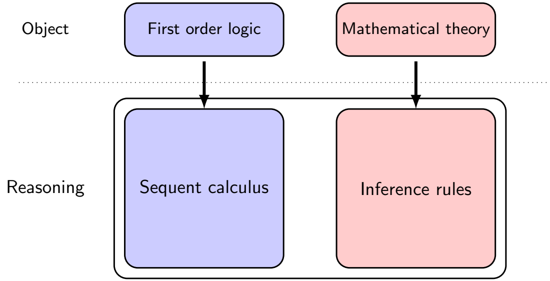

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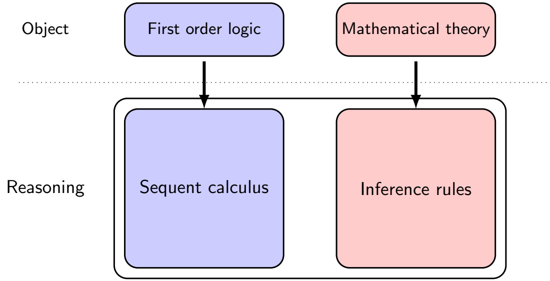

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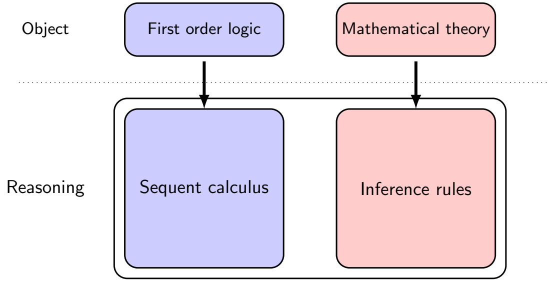

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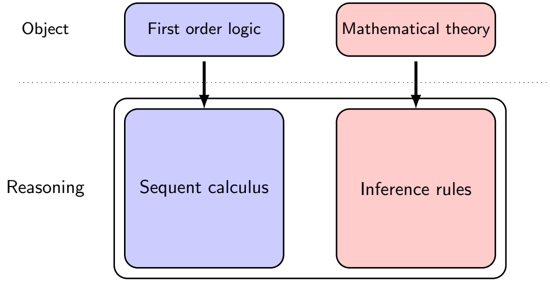

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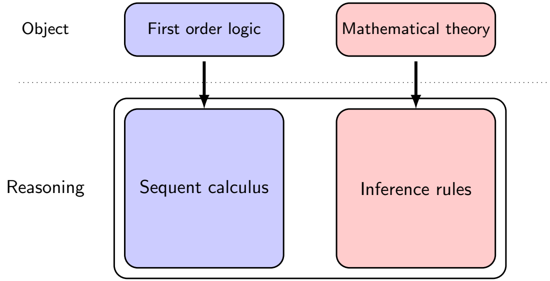

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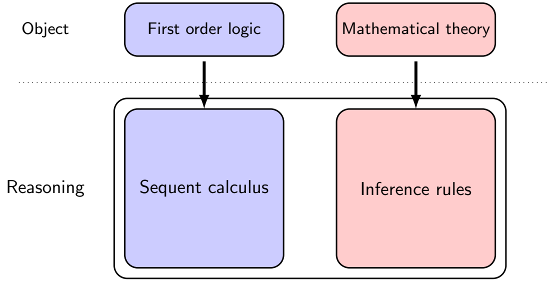

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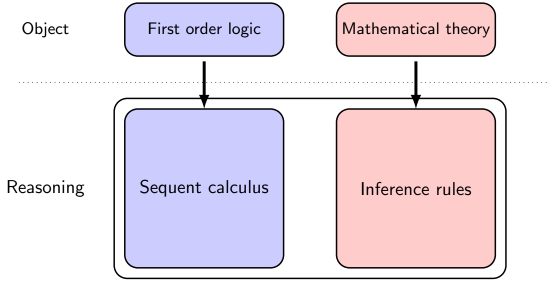

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- Systematically compute inference rules from bipolar axioms (λ-Prolog prototype);
- Uniform presentation for classical and intuitionistic first order systems;
- Generalisation of the literature (e.g. for Horn and geometric theories);
- Cut-elimination guaranteed for the system with the new inferences.

Outline

1. Polarities and bipolar formulas

2. Focusing and bipoles

3. Axioms-as-rules revisited

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First-order classical and intuitionistic language:

$$A ::= P(x) \mid A \wedge A \mid t \mid A \vee A \mid f \mid A \rightarrow A \mid \exists x\, A \mid \forall x\, A$$

Polarised connectives:

- In classical logic
  - positive and negative versions of the logical connectives and constants:
    $$\wedge^-, \wedge^+, t^-, t^+, \vee^-, \vee^+, f^-, f^+$$
  - first-order quantifiers: $\forall$ negative and $\exists$ positive.
- In intuitionistic logic
  - polarised classical connectives and constants where $f^-, \vee^-$ do not occur;
  - negative implication: $\rightarrow$.

Important: Even atomic formulas are polarised!

Hierarchy of negative and positive classical formulas. (Inspired by [Ciabattoni et al.])

$\mathcal{N}_0$ and $\mathcal{P}_0$ consist of all atoms

$$\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{N}_{n+1} \wedge^- \mathcal{N}_{n+1} \mid t^- \mid \mathcal{N}_{n+1} \vee^- \mathcal{N}_{n+1} \mid f^- \mid \forall x \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \to \mathcal{N}_{n+1}$$

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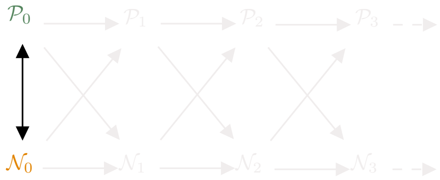
Q

R



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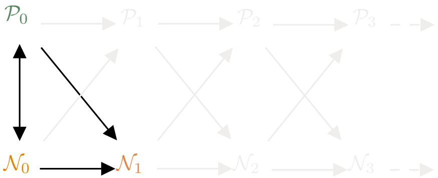
$Q_1 \wedge^- Q_2$



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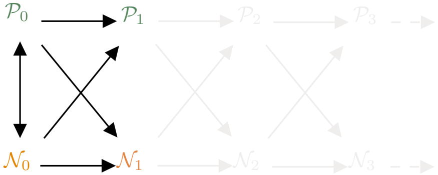

Polarity-based hierarchy

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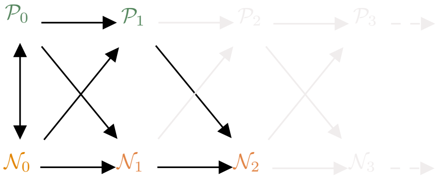
$(Q_1 \wedge^- Q_2) \to (R_1 \vee^+ R_2)$



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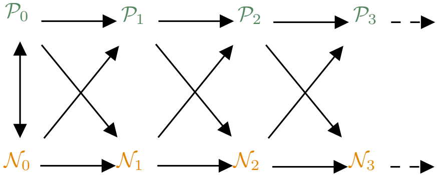

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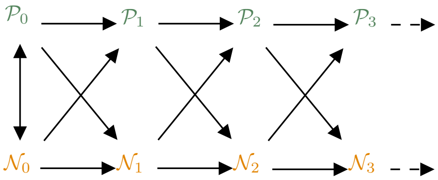

The hierarchy can be specified for intuitionistic or classical formulas.

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Aside: How to polarise a formula?

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Example. $(P_1 \to P_2) \vee (Q_1 \to Q_2) \leadsto$ classical bipolar $(P_1 \to P_2) \vee^- (Q_1 \to Q_2)$.

No polarisation yields an intuitionistic bipolar formula.

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What is focusing?

Consider the sequent

$$\Gamma, A_1 \to A_2 \to A_3 \to A_4 \to A_0 \vdash B$$

with A_i atomic, B a formula, and Γ a multiset of formulas.

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Focused rule application:
commit to repeat the $L\to$ rule on the right premise until the atomic formula A_0 results:

$$\dfrac{\Gamma \vdash A_1 \qquad \dfrac{\Gamma \vdash A_2 \qquad \dfrac{\Gamma \vdash A_3 \qquad \dfrac{\Gamma \vdash A_4 \qquad \Gamma, A_0 \vdash B}{\Gamma, A_4 \to A_0 \vdash B}\ L\to}{\Gamma, A_3 \to \cdots \to A_n \to A_0 \vdash B}\ L\to}{\Gamma, A_2 \to A_3 \to A_4 \to A_0 \vdash B}\ L\to}{\Gamma, A_1 \to A_2 \to A_3 \to A_4 \to A_0 \vdash B}\ L\to$$

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Unfocused

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- Chain together the other rules (non-invertible/consuming external information).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, B, \neg A} \quad \dfrac{}{A, B, \neg B}}{A, B, \neg A \wedge \neg B}\,\wedge}{A, B \vee C, \neg A \wedge \neg B}\,\vee}{\exists x.A, B \vee C, \neg A \wedge \neg B}\,\exists}{\exists x.A, \exists y.(B \vee C), \neg A \wedge \neg B}\,\exists}{\exists x.A, \exists y.(B \vee C), \forall z.(\neg A \wedge \neg B)}\,\forall$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A, \exists y.(B \vee C), \neg A}}{\exists x.A, \exists y.(B \vee C), \neg A}\,\exists \quad \dfrac{\dfrac{\dfrac{}{\exists x.A, B, \neg B}}{\exists x.A, B \vee C, \neg B}\,\vee}{\exists x.A, \exists y.(B \vee C), \neg B}\,\exists}{\exists x.A, \exists y.(B \vee C), \neg A \wedge \neg B}\,\wedge}{\exists x.A, \exists y.(B \vee C), \forall z.(\neg A \wedge \neg B)}\,\forall$$

Unfocused → Focused

Two kinds of focused sequents

- $\Downarrow$ sequents to decompose the formula under focus

$$\Gamma \Downarrow B \vdash \Delta \text{ with a left focus on } B$$
$$\Gamma \vdash B \Downarrow \Delta \text{ with a right focus on } B$$

When the conclusion of an introduction rule, then that rule introduced B.

- $\Uparrow$ sequents for invertible introduction rules

$$\Gamma_1 \Uparrow \Gamma_2 \vdash \Delta_1 \Uparrow \Delta_2$$

LKF and LJF

Two kinds of focused sequents

- $\Downarrow$ sequents to decompose the formula under focus

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  When the conclusion of an introduction rule, then that rule introduced B.

- $\Uparrow$ sequents for invertible introduction rules

$$\Gamma_1 \Uparrow \Gamma_2 \vdash \Delta_1 \Uparrow \Delta_2$$

$\Rightarrow$ Sequent derivations are organised into synchronous/asynchronous phases

$\Rightarrow$ Synthetic rules result from looking only at border sequents: $\Gamma \Uparrow \cdot \vdash \cdot \Uparrow \Delta$

Bipole

Let B be a polarised negative (classical or intuitionistic) formula.

A bipole for B is a synthetic inference rule corresponding to a derivation (in LKF or LJF)

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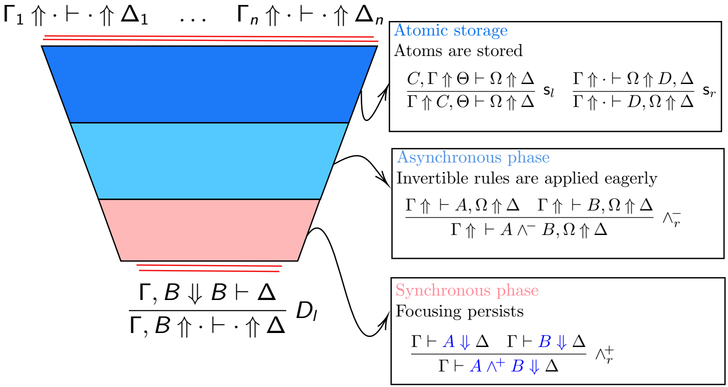

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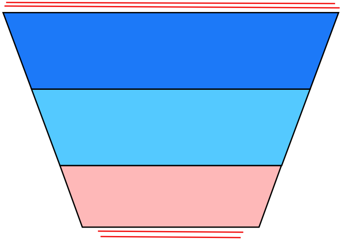
$$\Gamma_1 \Uparrow \cdot \vdash \cdot \Uparrow \Delta_1 \quad \ldots \quad \Gamma_n \Uparrow \cdot \vdash \cdot \Uparrow \Delta_n$$



$$\dfrac{\Gamma, B \Downarrow B \vdash \Delta}{\Gamma, B \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ D_l$$

Corresponding synthetic rule

(in LK or LJ)

$$\dfrac{\Gamma_1 \vdash \Delta_1 \quad \ldots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

Outline

1. Polarities and bipolar formulas

2. Focusing and bipoles

3. Axioms-as-rules revisited

Let B be a polarised negative (classical or intuitionistic) formula.

Theorem:

- If B is bipolar, then any synthetic inference rule for B is a bipole.
- If every synthetic inference rule for B is a bipole then B is bipolar.

This delineates precisely the scope of the relationship between axioms and rules!

▷ And provides the answer to Which ones?

Rules from axioms

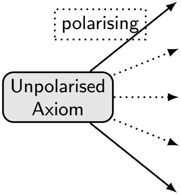
How?

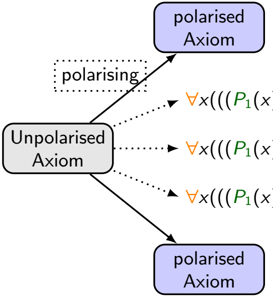
Rules from axioms

Unpolarised Axiom

$$\forall x(((P_1(x) \rightarrow P_2(x)) \wedge Q(x)) \rightarrow \exists y R(x, y))$$

Rules from axioms





$$\forall x(((P_1(x) \to P_2(x)) \wedge^+ Q(x)) \to \exists y R(x, y))$$

$$\forall x(((P_1(x) \to P_2(x)) \wedge^- Q(x)) \to \exists y R(x, y))$$

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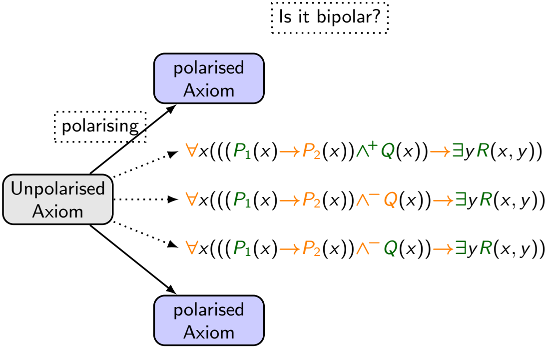
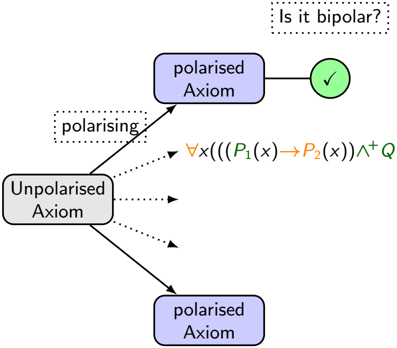
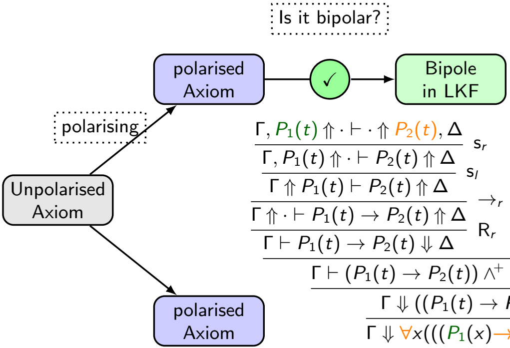
Rules from axioms



Rules from axioms

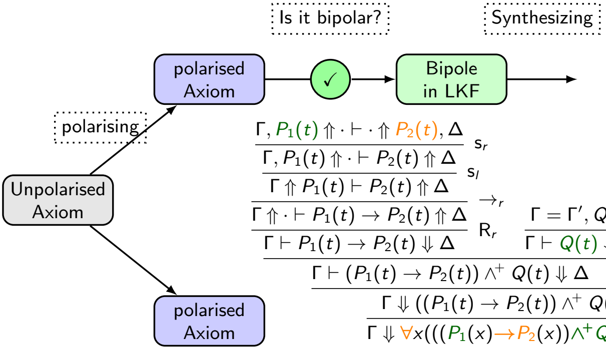


$\forall x(((P_1(x) \rightarrow P_2(x)) \wedge^+ Q(x)) \rightarrow \exists y R(x, y))$

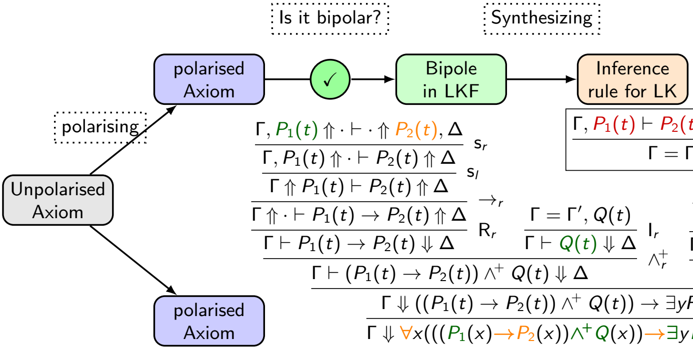


$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, P_1(t) \Uparrow \cdot \vdash \cdot \Uparrow P_2(t), \Delta}{\Gamma, P_1(t) \Uparrow \cdot \vdash P_2(t) \Uparrow \Delta}\,s_r}{\Gamma \Uparrow P_1(t) \vdash P_2(t) \Uparrow \Delta}\,s_l}{\Gamma \Uparrow \cdot \vdash P_1(t) \to P_2(t) \Uparrow \Delta}\,\to_r}{\Gamma \vdash P_1(t) \to P_2(t) \Downarrow \Delta}\,R_r \qquad \dfrac{\Gamma = \Gamma', Q(t)}{\Gamma \vdash Q(t) \Downarrow \Delta}\,I_r}{\Gamma \vdash (P_1(t) \to P_2(t)) \wedge^+ Q(t) \Downarrow \Delta}\,\wedge^+_r \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma, R(t,z) \Uparrow \cdot \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow R(t,z) \vdash \cdot \Uparrow \Delta}\,s_l}{\Gamma \Uparrow \exists y R(t,y) \vdash \cdot \Uparrow \Delta}\,\exists_l}{\Gamma \Downarrow \exists y R(t,y) \vdash \Delta}\,R_l}{\Gamma \Downarrow ((P_1(t) \to P_2(t)) \wedge^+ Q(t)) \to \exists y R(t,y) \vdash \Delta}\,\to_l}{\Gamma \Downarrow \forall x (((P_1(x) \to P_2(x)) \wedge^+ Q(x)) \to \exists y R(x,y)) \vdash \Delta}\,\forall_l$$

Rules from axioms



$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, P_1(t) \Uparrow \cdot \vdash \cdot \Uparrow P_2(t), \Delta}{\Gamma, P_1(t) \Uparrow \cdot \vdash P_2(t) \Uparrow \Delta}\, s_r}{\Gamma \Uparrow P_1(t) \vdash P_2(t) \Uparrow \Delta}\, s_l}{\Gamma \Uparrow \cdot \vdash P_1(t) \to P_2(t) \Uparrow \Delta}\, \to_r}{\Gamma \vdash P_1(t) \to P_2(t) \Downarrow \Delta}\, R_r \quad \dfrac{\Gamma = \Gamma', Q(t)}{\Gamma \vdash Q(t) \Downarrow \Delta}\, I_r}{\Gamma \vdash (P_1(t) \to P_2(t)) \wedge^+ Q(t) \Downarrow \Delta}\, \wedge^+_r \quad \dfrac{\dfrac{\dfrac{\Gamma, R(t,z) \Uparrow \cdot \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow R(t,z) \vdash \cdot \Uparrow \Delta}\, s_l}{\Gamma \Uparrow \exists y R(t,y) \vdash \cdot \Uparrow \Delta}\, \exists_l}{\Gamma \Downarrow \exists y R(t,y) \vdash \Delta}\, R_l}{\dfrac{\Gamma \Downarrow ((P_1(t) \to P_2(t)) \wedge^+ Q(t)) \to \exists y R(t,y) \vdash \Delta}{\Gamma \Downarrow \forall x(((P_1(x) \to P_2(x)) \wedge^+ Q(x)) \to \exists y R(x,y)) \vdash \Delta}\, \forall_l}\, \to_l$$



$$\frac{\Gamma, P_1(t) \vdash P_2(t), \Delta \qquad \Gamma, R(t,z) \vdash \Delta}{\Gamma = \Gamma', Q(t) \vdash \Delta}$$

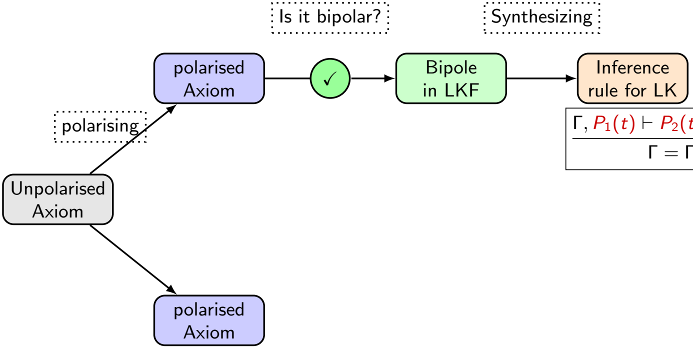
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, P_1(t) \Uparrow \cdot \vdash \cdot \Uparrow P_2(t), \Delta}{\Gamma, P_1(t) \Uparrow \cdot \vdash P_2(t) \Uparrow \Delta}\, s_r}{\Gamma \Uparrow P_1(t) \vdash P_2(t) \Uparrow \Delta}\, s_l}{\Gamma \Uparrow \cdot \vdash P_1(t) \to P_2(t) \Uparrow \Delta}\, \to_r}{\Gamma \vdash P_1(t) \to P_2(t) \Downarrow \Delta}\, R_r \qquad \dfrac{\Gamma = \Gamma', Q(t)}{\Gamma \vdash Q(t) \Downarrow \Delta}\, I_r$$

$$\dfrac{\Gamma \vdash P_1(t) \to P_2(t) \Downarrow \Delta \qquad \Gamma \vdash Q(t) \Downarrow \Delta}{\Gamma \vdash (P_1(t) \to P_2(t)) \wedge^+ Q(t) \Downarrow \Delta}\, \wedge^+_r$$

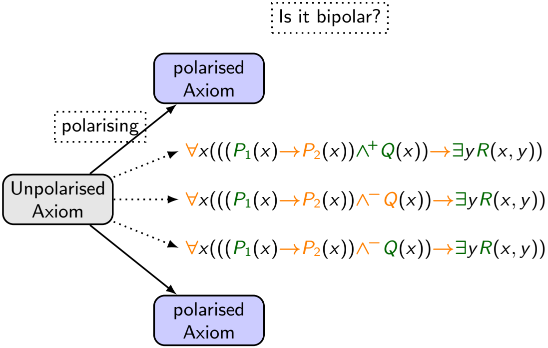
$$\dfrac{\dfrac{\dfrac{\Gamma, R(t,z) \Uparrow \cdot \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow R(t,z) \vdash \cdot \Uparrow \Delta}\, s_l}{\Gamma \Uparrow \exists y R(t,y) \vdash \cdot \Uparrow \Delta}\, \exists_l}{\Gamma \Downarrow \exists y R(t,y) \vdash \Delta}\, R_l$$

$$\dfrac{\dfrac{\Gamma \vdash (P_1(t) \to P_2(t)) \wedge^+ Q(t) \Downarrow \Delta \qquad \Gamma \Downarrow \exists y R(t,y) \vdash \Delta}{\Gamma \Downarrow ((P_1(t) \to P_2(t)) \wedge^+ Q(t)) \to \exists y R(t,y) \vdash \Delta}\, \to_l}{\Gamma \Downarrow \forall x (((P_1(x) \to P_2(x)) \wedge^+ Q(x)) \to \exists y R(x,y)) \vdash \Delta}\, \forall_l$$

Rules from axioms



$$\frac{\Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma, R(t,z) \vdash \Delta}{\Gamma = \Gamma', Q(t) \vdash \Delta}$$


Is it bipolar?
polarised Axiom
polarising
$\forall x(((P_1(x) \rightarrow P_2(x)) \wedge^+ Q(x)) \rightarrow \exists y R(x,y))$
Unpolarised Axiom
$\forall x(((P_1(x) \rightarrow P_2(x)) \wedge^- Q(x)) \rightarrow \exists y R(x,y))$
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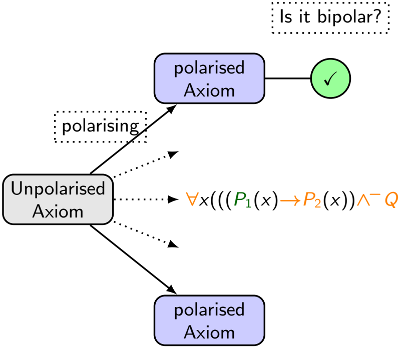
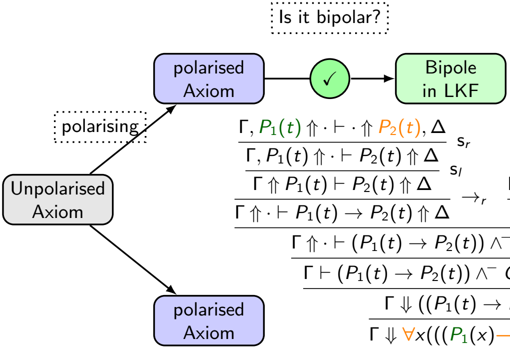
polarised Axiom

Rules from axioms



$\forall x(((P_1(x) \rightarrow P_2(x)) \wedge^- Q(x)) \rightarrow \exists y R(x, y))$

Rules from axioms



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\dfrac{\Gamma, P_1(t) \Uparrow \cdot \vdash \cdot \Uparrow P_2(t), \Delta}{\Gamma, P_1(t) \Uparrow \cdot \vdash P_2(t) \Uparrow \Delta}\ s_r
}{\Gamma \Uparrow P_1(t) \vdash P_2(t) \Uparrow \Delta}\ s_l
}{\Gamma \Uparrow \cdot \vdash P_1(t) \to P_2(t) \Uparrow \Delta}\ \to_r
\qquad
\dfrac{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow Q(t), \Delta}{\Gamma \Uparrow \cdot \vdash Q(t) \Uparrow \Delta}\ s_r
}{\Gamma \Uparrow \cdot \vdash (P_1(t) \to P_2(t)) \wedge^- Q(t) \Uparrow \Delta}\ \wedge_r^-
}{\Gamma \vdash (P_1(t) \to P_2(t)) \wedge^- Q(t) \Downarrow \Delta}\ R_r
\qquad
\dfrac{
\dfrac{
\dfrac{\Gamma, R(t,z) \Uparrow \cdot \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow R(t,z) \vdash \cdot \Uparrow \Delta}\ s_l
}{\Gamma \Uparrow \exists y R(t,y) \vdash \cdot \Uparrow \Delta}\ \exists_l
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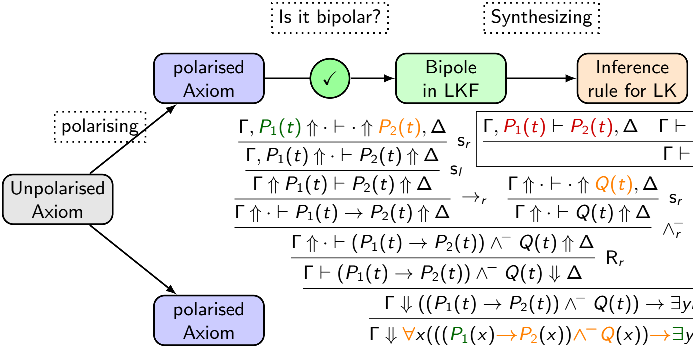
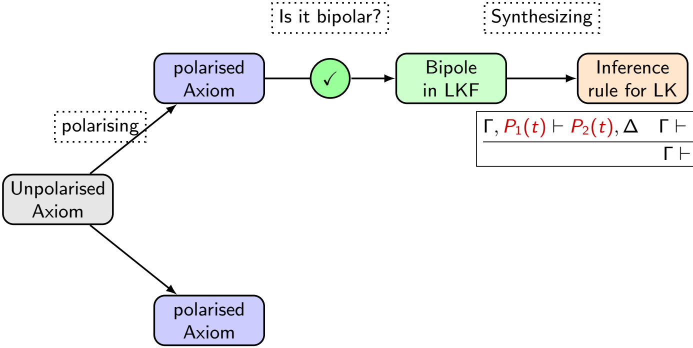
$$

Rules from axioms



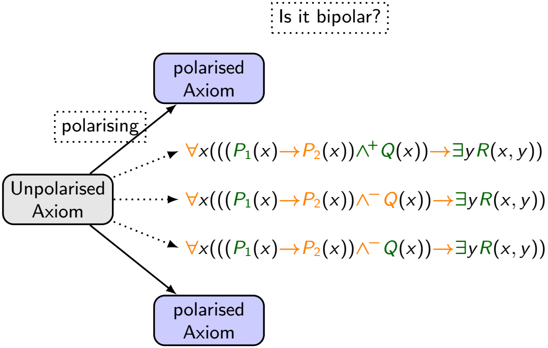
$$\dfrac{\Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma \vdash Q(t), \Delta \quad \Gamma, R(t,z) \vdash \Delta}{\Gamma \vdash \Delta}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, P_1(t) \Uparrow \cdot \vdash \cdot \Uparrow P_2(t), \Delta}{\Gamma, P_1(t) \Uparrow \cdot \vdash P_2(t) \Uparrow \Delta}\, s_r}{\Gamma \Uparrow P_1(t) \vdash P_2(t) \Uparrow \Delta}\, s_l}{\Gamma \Uparrow \cdot \vdash P_1(t) \to P_2(t) \Uparrow \Delta}\, \to_r \quad \dfrac{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow Q(t), \Delta}{\Gamma \Uparrow \cdot \vdash Q(t) \Uparrow \Delta}\, s_r}{\dfrac{\Gamma \Uparrow \cdot \vdash (P_1(t) \to P_2(t)) \wedge^- Q(t) \Uparrow \Delta}{\Gamma \vdash (P_1(t) \to P_2(t)) \wedge^- Q(t) \Downarrow \Delta}\, R_r}\, \wedge^-_r \quad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma, R(t,z) \Uparrow \cdot \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow R(t,z) \vdash \cdot \Uparrow \Delta}\, s_l}{\Gamma \Uparrow \exists y R(t,y) \vdash \cdot \Uparrow \Delta}\, \exists_l}{\Gamma \Downarrow \exists y R(t,y) \vdash \Delta}\, R_l}{\dfrac{\Gamma \Downarrow ((P_1(t) \to P_2(t)) \wedge^- Q(t)) \to \exists y R(t,y) \vdash \Delta}{\Gamma \Downarrow \forall x(((P_1(x) \to P_2(x)) \wedge^- Q(x)) \to \exists y R(x,y)) \vdash \Delta}\, \forall_l}\, \to_l$$



$$\frac{\Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma \vdash Q(t), \Delta \quad \Gamma, R(t,z) \vdash \Delta}{\Gamma \vdash \Delta}$$

Rules from axioms



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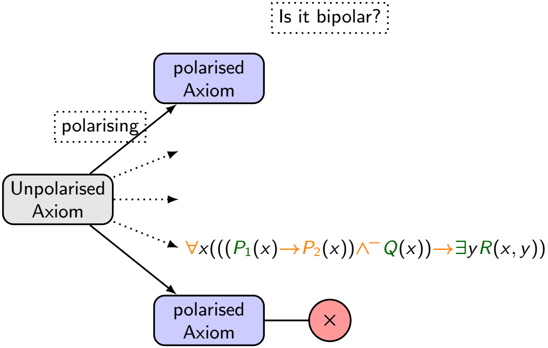
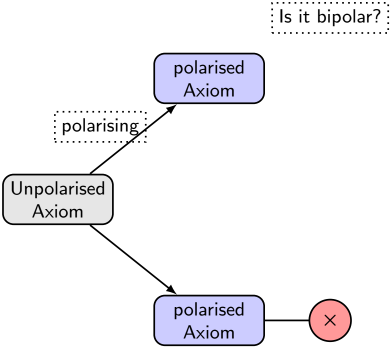
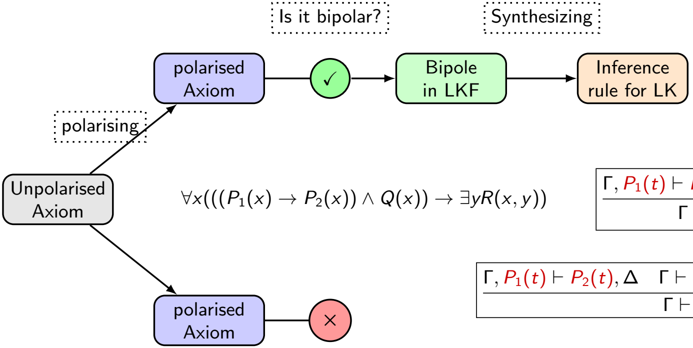


Rules from axioms





$$\forall x(((P_1(x) \to P_2(x)) \wedge Q(x)) \to \exists y R(x,y))$$

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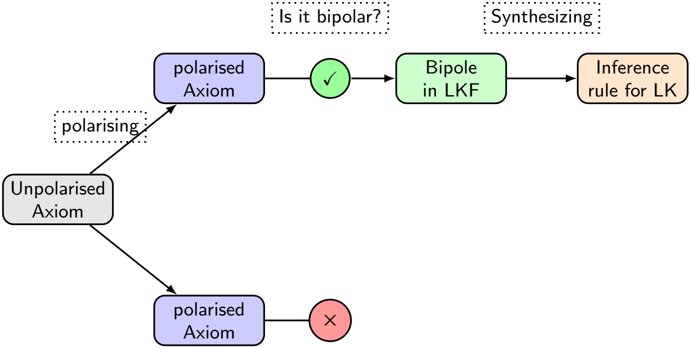
Rules from axioms



2nd result: Cut admissibility

Let $\mathcal{T}$ be a set of bipolar formulas.

LK$\langle\mathcal{T}\rangle$/LJ$\langle\mathcal{T}\rangle$ denotes the extension of LK/LJ with the synthetic inference rules corresponding to a bipole for each $B \in \mathcal{T}$.

Theorem: The cut rule is admissible for the proof systems LK$\langle\mathcal{T}\rangle$/LJ$\langle\mathcal{T}\rangle$.

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Note: the proof is simple!

It is a direct consequence of (polarised) cut admissibility in LKF/LJF.

$$\frac{\Gamma \Uparrow \cdot \vdash B \Uparrow \Delta \qquad \Gamma \Uparrow B \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ Cut$$

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This is why the obtained rules live in harmony within the sequent framework.

Back to examples.

Horn clauses as bipoles.

$$\forall \overline{z}(P_1 \wedge \ldots \wedge P_m \ \rightarrow \ Q)$$

Geometric axioms as bipoles.

$$\forall \overline{z}(P_1 \wedge \ldots \wedge P_m \ \rightarrow \ \exists \overline{x}_1(\wedge Q_1) \vee \ldots \vee \exists \overline{x}_n(\wedge Q_n))$$

Back to examples.

Horn clauses as bipoles.

$$\forall \overline{z}(P_1^{\pm} \wedge^{\pm} \ldots \wedge^{\pm} P_m^{\pm} \rightarrow Q^{\pm})$$

Geometric axioms as bipoles.

$$\forall \overline{z}(P_1^{\pm} \wedge^{\pm} \ldots \wedge^{\pm} P_m^{\pm} \rightarrow \exists \overline{x}_1(\wedge^{+} Q_1^{\pm}) \vee^{\pm} \ldots \vee^{\pm} \exists \overline{x}_n(\wedge^{+} Q_n^{\pm}))$$

Back to examples.

Horn clauses as bipoles.

$$\forall \overline{z}(P_1^{\pm} \wedge^{\pm} \ldots \wedge^{\pm} P_m^{\pm} \rightarrow Q^{\pm})$$

$$\forall \overline{z}(P_1^{+} \wedge^{+} \ldots \wedge^{+} P_m^{+} \rightarrow Q^{+})$$

$$\frac{Q, \Gamma \vdash C}{P_1, \ldots, P_m, \Gamma \vdash C} \; (\textit{forward})$$

Geometric axioms as bipoles.

$$\forall \overline{z}(P_1^{\pm} \wedge^{\pm} \ldots \wedge^{\pm} P_m^{\pm} \rightarrow \exists \overline{x}_1(\wedge^{+} Q_1^{\pm}) \vee^{\pm} \ldots \vee^{\pm} \exists \overline{x}_n(\wedge^{+} Q_n^{\pm}))$$

Back to examples.

Horn clauses as bipoles.

$$\forall\overline{z}(P_1^{\pm} \wedge^{\pm} \ldots \wedge^{\pm} P_m^{\pm} \to Q^{\pm})$$

$$\forall\overline{z}(P_1^{+} \wedge^{+} \ldots \wedge^{+} P_m^{+} \to Q^{+}) \qquad \forall\overline{z}(P_1^{-} \wedge^{-} \ldots \wedge^{-} P_m^{-} \to Q^{-})$$

$$\frac{Q, \Gamma \vdash C}{P_1, \ldots, P_m, \Gamma \vdash C}\ (\mathit{forward}) \qquad \frac{\Gamma \vdash P_1, \quad \ldots \quad \Gamma \vdash P_m}{\Gamma \vdash Q}\ (\mathit{back})$$

Geometric axioms as bipoles.

$$\forall\overline{z}(P_1^{\pm} \wedge^{\pm} \ldots \wedge^{\pm} P_m^{\pm} \to \exists\overline{x}_1(\wedge^{+} Q_1^{\pm}) \vee^{\pm} \ldots \vee^{\pm} \exists\overline{x}_n(\wedge^{+} Q_n^{\pm}))$$

Back to examples.

Horn clauses as bipoles.

$$\forall \overline{z}(P_1^{\pm} \wedge^{\pm} \ldots \wedge^{\pm} P_m^{\pm} \to Q^{\pm})$$

$$\forall \overline{z}(P_1^{+} \wedge^{+} \ldots \wedge^{+} P_m^{+} \to Q^{+}) \qquad \forall \overline{z}(P_1^{-} \wedge^{-} \ldots \wedge^{-} P_m^{-} \to Q^{-})$$

$$\frac{Q, \Gamma \vdash C}{P_1, \ldots, P_m, \Gamma \vdash C}\ (\mathit{forward}) \qquad \frac{\Gamma \vdash P_1, \quad \ldots \quad \Gamma \vdash P_m}{\Gamma \vdash Q}\ (\mathit{back})$$

Geometric axioms as bipoles.

$$\forall \overline{z}(P_1^{+} \wedge^{+} \ldots \wedge^{+} P_m^{+} \to \exists \overline{x}_1(\wedge^{+} Q_1^{\pm}) \vee^{\pm} \ldots \vee^{\pm} \exists \overline{x}_n(\wedge^{+} Q_n^{\pm}))$$

$$\frac{\overline{Q}_1, \Gamma \vdash C \quad \ldots \quad \overline{Q}_n, \Gamma \vdash C}{P_1, \ldots, P_m, \Gamma' \vdash C}\ (\mathit{geom})$$

Some discussion

Logics on first-order structures.

Modal and substructural logics in labelled sequents. [Simpson,Viganò,Ciabattoni,. . .]

- Direct consequence of synthetisation result.

Logics on first-order structures.

Modal and substructural logics in labelled sequents. [Simpson,Viganò,Ciabattoni,. . .]

- Direct consequence of synthetisation result.

Arbitrary first-order axioms.

- System of rules: for generalised geometric formulas. [Negri]
  - Our approach should apply and generalise this to "generalised bipolar axioms".
- Morleyisation: every first-order theory is equivalent to a geometric theory. Adapted to the proof theoretic setting. [Dyckhoff and Negri]
  - We need to study how the polarisation of formulas commute with their "geometrisation".

To conclude

Axiom + Focusing = Rules

- ⋆ Synthetic inference rules generated using polarisation and focusing provide inference rules that capture certain classes of axioms.
- ⋆ In particular: bipolar formulas correspond to inference rules for atoms.
- ⋆ As geometric formulas are examples of bipolar formulas, polarised versions of such formulas yield well known inference systems derived from geometric formulas.
- ⋆ Polarisation of subsets of geometric formulas explain the forward-chaining and backward-chaining variants of their synthetic inference rules.
- ⋆ Direct proof of cut-elimination for the proof systems that arise from incorporating synthetic inference rules based on polarised formulas.
- ⋆ Additionally, all of these results work equally well in both classical and intuitionistic logics using the corresponding LKF and LJF focused proof systems.
- ⋆ Future work: beyond focusing-bipoles.

Thank you!