A cut-free proof system for pseudo-transitive modal logics

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Classical Modal Logic

- Formulas: $A, B, \ldots ::= p \mid \bar{p} \mid A \wedge B \mid A \vee B \mid \Box A \mid \Diamond A$
- Axioms for K: classical propositional logic and

$$\mathsf{k}\colon \Box(A \to B) \to (\Box A \to \Box B)$$

- Rules: modus ponens: $\dfrac{A \quad A \to B}{B}$ necessitation: $\dfrac{A}{\Box A}$

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- Theorem: The logic K is sound and complete wrt Kripke frames $\langle W, R\rangle$.
  - W a non-empty set of *worlds*
  - $R \subseteq W \times W$ the *accessibility* relation

A fine selection of modal axioms

t: $\Box A \to A$	reflexivity	$\forall w. wRw$
4: $\Box A \to \Box\Box A$	transitivity	$\forall xyw. xRy \wedge yRw \to xRw$
4^*: $\Box\Box A \to \Box\Box\Box A$	pseudo-transitivity	$\forall xyzw. xRy \wedge yRz \wedge zRw \to \exists u. xRu \wedge uRw$

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4^n_m: $\Box^n A \to \Box^m A$	(m,n)-transitivity	$\forall xw.xR^m w \to xR^n w$

Pure nested sequents

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- Sequent context: $\Gamma\{\ \}\{\ \}\{\ \} = A, [\{\ \}], [B, \{\ \}, [\{\ \}]]$
- System KN:

$$\mathsf{id}\ \frac{}{\Gamma\{a, \bar{a}\}} \quad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \quad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \quad \Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \quad \Diamond\ \frac{\Gamma\{\Diamond A, [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

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- Theorem: *System* KN *is sound and complete for the logic* K.

[Kashima, 1994], [Brünnler, 2009], [Poggiolesi, 2009]

Indexed nested sequents

- Indexed Sequent: $\Gamma ::= A_1, \ldots, A_m, [^{w_1}\Gamma_1], \ldots, [^{w_n}\Gamma_n]$
- No corresponding formula in the general case
- Indexed context: $\Gamma\,{}^2\{\ \}\,{}^1\{\ \}\,{}^2\{\ \} = A, [^2\{\ \}], [^1 B, \{\ \}, [^2\{\ \}]]$
- System iKN:

$$\mathsf{id}\ \frac{}{\Gamma\{a, \bar{a}\}} \quad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \quad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \quad \Box\ \frac{\Gamma\{[^v A]\}}{\Gamma\{\Box A\}} \quad \Diamond\ \frac{\Gamma\{\Diamond A, [^u A, \Delta]\}}{\Gamma\{\Diamond A, [^u \Delta]\}}$$

[Fitting, 2015]

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$$\mathsf{tp}\,\frac{\Gamma\,^w\{\emptyset\}\,^w\{A\}}{\Gamma\,^w\{A\}\,^w\{\emptyset\}} \qquad \mathsf{bc}\,\frac{\Gamma\,^w\{[^u\Delta]\}\,^w\{[^u\ ]\}}{\Gamma\,^w\{[^u\Delta]\}\,^w\{\emptyset\}}$$

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- Theorem: *System* iKN *is sound and complete for the logic* K.

[Fitting, 2015]

Why are we doing this?

$\mathsf{t}\colon \Box A \to A$ $\quad\quad$ $\dot{\mathsf{t}}\ \dfrac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}}$

$\mathsf{4}\colon \Box A \to \Box\Box A$ $\quad\quad$ $\dot{\mathsf{4}}\ \dfrac{\Gamma\{[\Delta]\}}{\Gamma\{[[\Delta]]\}}$

Theorem: *System* $\mathsf{KN} + \dot{\mathsf{t}}$ *is sound and complete for the logic* $\mathsf{K} + \mathsf{t}$.

Problem: *System* $\mathsf{KN} + \dot{\mathsf{4}}$ *is not complete for the logic* $\mathsf{K} + \mathsf{4}$.

Why are we doing this?

$\mathsf{t}\colon \Box A \to A$ $\quad \dot{\mathsf{t}}\ \dfrac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}}$ $\quad \mathsf{t}^{\Diamond}\ \dfrac{\Gamma\{\Diamond A, A\}}{\Gamma\{\Diamond A\}}$

$4\colon \Box A \to \Box\Box A$ $\quad \dot{4}\ \dfrac{\Gamma\{[\Delta]\}}{\Gamma\{[[\Delta]]\}}$ $\quad 4^{\Diamond}\ \dfrac{\Gamma\{\Diamond A, [\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$

Theorem: *System* $\mathsf{KN} + \mathsf{t}^{\Diamond}$ *is sound and complete for the logic* $\mathsf{K} + \mathsf{t}$.

Theorem: *System* $\mathsf{KN} + 4^{\Diamond}$ *is sound and complete for the logic* $\mathsf{K} + 4$.

Why are we doing this?

$\mathsf{t}\colon \Box A \to A$ $\qquad \dot{\mathsf{t}}\ \dfrac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}}$ $\qquad \mathsf{t}^\Diamond\ \dfrac{\Gamma\{\Diamond A, A\}}{\Gamma\{\Diamond A\}}$

$4\colon \Box A \to \Box\Box A$ $\qquad \dot{4}\ \dfrac{\Gamma\{[\Delta]\}}{\Gamma\{[[\Delta]]\}}$ $\qquad 4^\Diamond\ \dfrac{\Gamma\{\Diamond A, [\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$

$4^*\colon \Box\Box A \to \Box\Box\Box A$ $\qquad \dot{4}^*\ \dfrac{\Gamma\{[[\Delta]]\}}{\Gamma\{[[[\Delta]]]\}}$ $\qquad$ no $\Diamond$-rule!

Theorem: *System* $\mathsf{KN} + \mathsf{t}^\Diamond$ *is sound and complete for the logic* $\mathsf{K} + \mathsf{t}$.

Theorem: *System* $\mathsf{KN} + 4^\Diamond$ *is sound and complete for the logic* $\mathsf{K} + 4$.

Problem: complete system for $\mathsf{K} + 4^*$?

Why are we doing this?

$\mathsf{t}\colon \Box A \to A$

$$\dot{4}^1_0 \; \frac{\Gamma^{w}\{[^{w}\;]\}}{\Gamma^{w}\{\emptyset\}}$$

$4\colon \Box A \to \Box\Box A$

$$\dot{4}^1_2 \; \frac{\Gamma\{[^{w}\;], [^{x}[^{w}\Delta_1], \Delta_2]\}}{\Gamma\{[^{x}[^{w}\Delta_1], \Delta_2]\}}$$

$\mathsf{4}^{\mathsf{n}}_{\mathsf{m}}\colon \Box^n A \to \Box^m A$

$$\dot{4}^n_m \; \frac{\Gamma\{[^{u_n}\cdots[^{u_2}[^{w}\;]]], [^{x_m}\cdots[^{x_2}[^{w}\Delta_1], \Delta_2], \cdots\Delta_m]\}}{\Gamma\{[^{x_m}\cdots[^{x_2}[^{w}\Delta_1], \Delta_2], \cdots\Delta_m]\}}$$

Theorem: *System* $\mathsf{iKN} + \dot{4}^n_m$ *is sound and complete for the logic* $\mathsf{K} + 4^n_m$.

Proof via syntactic cut-elimination

Theorem: *If a sequent is derivable in* $\mathsf{iNK} + \dot{4}^n_m$ *together with the* cut*-rule*

$$\mathsf{cut}\,\frac{\Gamma\{A\} \quad \Gamma\{\bar{A}\}}{\Gamma\{\emptyset\}}$$

then it is also derivable in $\mathsf{iNK} + \dot{4}^n_m$ *without* cut.

Ongoing work

- decidability problems
- extension to a larger selection of modal axioms
- extension to intuitionistic modal logics
- ...

Kripke semantics

- Kripke model $\mathcal{M} = \langle W, R, V \rangle$:
  - a non-empty set W of *worlds*
  - an *accessibility relation* $R \subseteq W \times W$,
  - a *valuation* function $V\colon W \to 2^{\mathcal{A}}$
-

$$
\begin{array}{lll}
w \Vdash p & \text{iff} & w \in V(p) \\
w \Vdash \bar{p} & \text{iff} & w \nVdash p \\
w \Vdash A \wedge B & \text{iff} & w \Vdash A \text{ and } w \Vdash B \\
w \Vdash A \vee B & \text{iff} & w \Vdash A \text{ or } w \Vdash B \\
w \Vdash \Box A & \text{iff} & \text{for all } w' \text{ if } w'Ru\text{, we have } u \Vdash A \\
w \Vdash \Diamond A & \text{iff} & \text{there is a } u \in W \text{ such that } wRu \text{ and } u \Vdash A
\end{array}
$$

$$
\begin{array}{r l}
 & \Diamond\Box q, [q], [[q], q, p], [[q, p], q, \bar{p}, \Diamond p], [[q, \bar{p}, \Diamond p], \Diamond\bar{p}, \Diamond\Diamond p] \\ \cline{2-2}
\text{cont}, \dot{4} & \Diamond\Box q, [[q], q, p], [[q, p], q, \bar{p}, \Diamond p], [[q, \bar{p}, \Diamond p], \Diamond\bar{p}, \Diamond\Diamond p] \\ \cline{2-2}
\Box & \Diamond\Box q, [\Box q, q, p], [[q, p], q, \bar{p}, \Diamond p], [[q, \bar{p}, \Diamond p], \Diamond\bar{p}, \Diamond\Diamond p] \\ \cline{2-2}
\text{cont}, \Diamond & \Diamond\Box q, [q, p], [[q, p], q, \bar{p}, \Diamond p], [[q, \bar{p}, \Diamond p], \Diamond\bar{p}, \Diamond\Diamond p] \\ \cline{2-2}
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\Box & \Diamond\Box q, [\Box q, q, \bar{p}, \Diamond p], [[q, \bar{p}, \Diamond p], \Diamond\bar{p}, \Diamond\Diamond p] \\ \cline{2-2}
\text{cont}, \Diamond & \Diamond\Box q, [q, \bar{p}, \Diamond p], [[q, \bar{p}, \Diamond p], \Diamond\bar{p}, \Diamond\Diamond p] \\ \cline{2-2}
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\Box & \Diamond\Box q, [\Box q, \Diamond\bar{p}, \Diamond\Diamond p] \\ \cline{2-2}
\text{cont}, \Diamond & \Diamond\Box q, [\Diamond\bar{p}, \Diamond\Diamond p] \\ \cline{2-2}
\vee, \Box, \vee & \Diamond\Box q \vee \Box(\Diamond\bar{p} \vee \Diamond\Diamond p)
\end{array}
$$