Focused and Synthetic Nested Sequents

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Classical modal logic

Formulas: $A ::= a \mid \bar{a} \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A$

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Axioms for K**:** classical propositional logic and

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Rules: modus ponens: $\dfrac{A \quad A \to B}{B}$ necessitation: $\dfrac{A}{\Box A}$

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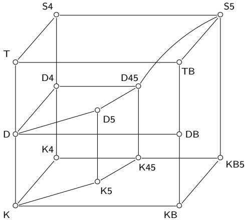
The S5**-cube:**

$$\begin{aligned} \mathsf{d}&\colon \ \Box A \to \Diamond A \\ \mathsf{t}&\colon \ A \to \Diamond A \\ \mathsf{b}&\colon \ A \to \Box\Diamond A \\ 4&\colon \ \Diamond\Diamond A \to \Diamond A \\ 5&\colon \ \Diamond A \to \Box\Diamond A \end{aligned}$$



Nested sequents

Sequent: $$\Gamma ::= A_1, \ldots, A_m$$

$$fm(\Gamma) = A_1 \vee \ldots \vee A_m$$

Nested sequent: $\Gamma ::= A_1, \ldots, A_m, [\Gamma_1], \ldots, [\Gamma_n]$

$$fm(\Gamma) = A_1 \vee \ldots \vee A_m \vee \Box fm(\Gamma_1) \vee \ldots \vee \Box fm(\Gamma_n)$$

$$\begin{tikzcd}[column sep=small] & A_1, \ldots, A_m \arrow[dl, dash] \arrow[dr, dash] & \\ \Gamma_1 & \cdots & \Gamma_n \end{tikzcd}$$

[Kashima, 1994], [Brünnler, 2009], [Poggiolesi, 2009]

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Sequent context: $\Gamma\{\ \}\{\ \} = A, B, [C, [\{\ \}]], [D, \{\ \}]$

$$\begin{array}{ccc} & A, B & \\ C & & D, \{\ \} \\ \{\ \} & & \end{array}$$

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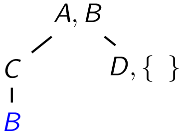

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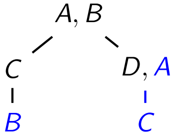

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The standard nested system

Formulas: $A \quad ::= \quad a \mid \bar{a} \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A$

System KN**:**

$$\mathsf{cont}\,\frac{\Gamma\{A, A\}}{\Gamma\{A\}} \qquad \Box\,\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \vee\,\frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

$$\mathsf{id}\,\frac{}{\Gamma\{a, \bar{a}\}} \qquad \mathsf{k}^{\Diamond}\,\frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad \wedge\,\frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}$$

k: $\Box(A \to B) \to (\Box A \to \Box B)$

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$\mathsf{k}\colon \Box(A \to B) \to (\Box A \to \Box B)$

Modal rules:

$$\mathsf{d}^{\Diamond}\ \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{t}^{\Diamond}\ \frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{b}^{\Diamond}\ \frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Diamond A]\}} \qquad \mathsf{4}^{\Diamond}\ \frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad \mathsf{5}^{\Diamond}\ \frac{\Gamma\{\emptyset\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}$$

$\mathsf{d}\colon \Box A \to \Diamond A$ $\quad$ $\mathsf{t}\colon A \to \Diamond A$ $\quad$ $\mathsf{b}\colon A \to \Box\Diamond A$ $\quad$ $\mathsf{4}\colon \Diamond\Diamond A \to \Diamond A$ $\quad$ $\mathsf{5}\colon \Diamond A \to \Box\Diamond A$

[Brünnler, 2009]

Polarity and focusing

Polarities:	Negative connectives	:	invertible rules
	Positive connectives	:	non-invertible rules

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Weak focusing: For any subproof $\begin{array}{c} {\scriptstyle \pi'} \Big\Vert \\ \Gamma\{P\} \end{array}$ the only positive rules between two rules decomposing P are rules decomposing P.

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Strong focusing: For any subproof $\begin{array}{c} {\scriptstyle \pi'} \Big\| \\ \Gamma\{P\} \end{array}$ the only rules between two rules decomposing P are rules decomposing P.

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Strong focusing: For any subproof $\begin{array}{c} \pi' \Big\| \\ \Gamma\{P\} \end{array}$ the only rules between two rules decomposing P are rules decomposing P.

Inversion: For any subproof $\begin{array}{c} \pi' \Big\| \\ \Gamma\{N\} \end{array}$ the last rule is negative.

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Modal rules:

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The focused nested system

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Polarized formulas:

$$P ::= a \mid {\downarrow}N \mid \Diamond P \mid P \wedge P$$
$$N ::= \bar{a} \mid {\uparrow}P \mid \Box N \mid N \vee N$$

System KN:

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The focused nested system

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Focused system KNF**:**

$$\mathsf{dec}\ \frac{\Gamma\{P, \langle P \rangle\}}{\Gamma\{P\}} \qquad \Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

$$\mathsf{id}\ \frac{}{\Gamma\{\bar{a}, \langle a \rangle\}} \qquad \mathsf{k}^{\Diamond}\ \frac{\Gamma\{[\langle A \rangle, \Delta]\}}{\Gamma\{\langle \Diamond A \rangle, [\Delta]\}} \qquad \wedge\ \frac{\Gamma\{\langle A \rangle\} \quad \Gamma\{\langle B \rangle\}}{\Gamma\{\langle A \wedge B \rangle\}}$$

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Modal rules:

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The focused nested system

Polarized formulas:

$$
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P & ::= & a \mid \downarrow N \mid \Diamond P \mid P \stackrel{+}{\wedge} P \mid P \stackrel{+}{\vee} P \\
N & ::= & \bar{a} \mid \uparrow P \mid \Box N \mid N \bar{\vee} N \mid N \bar{\wedge} N
\end{array}
$$

Focused system KNF**:**

$$
\text{dec}\ \frac{\Gamma\{P, \langle P\rangle\}}{\Gamma\{P\}} \qquad \Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \bar{\vee}\ \frac{\Gamma\{A, B\}}{\Gamma\{A \bar{\vee} B\}} \qquad \bar{\wedge}\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \bar{\wedge} B\}} \qquad \text{sto}\ \frac{\Gamma\{P\}}{\Gamma\{\uparrow P\}}
$$

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Focused modal rules:

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Completeness of focusing

Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$.
If A is provable in $\mathsf{KN} + \mathsf{X}^{\diamond}$, then any $\mathrm{pol}(A)$ is provable in $\mathsf{KNF} + \mathsf{X}^{\diamond}$.

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Via cut-elimination:

$$\mathsf{cut}\ \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

$$\mathsf{KN} \xrightarrow{\text{simulation}} \mathsf{KNF} + \mathsf{cut}$$

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Problem: weakening on negative formula!

$$\mathsf{cut}\,\dfrac{\mathsf{rel}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma\{N\}\{P\}}}{\Gamma\{\langle{\downarrow}N\rangle\}\{P\}} \quad \overset{\mathcal{D}_2}{\Gamma\{\emptyset\}\{\bar{P}\}}}{\Gamma\{\langle{\downarrow}N\rangle\}\{\emptyset\}} \quad \rightsquigarrow \quad \mathsf{rel}\,\dfrac{\mathsf{cut}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma\{N\}\{P\}} \quad \mathsf{weak}\,\dfrac{\overset{\mathcal{D}_2}{\Gamma\{\emptyset\}\{\bar{P}\}}}{\Gamma\{N\}\{\bar{P}\}}}{\Gamma\{N\}\{\emptyset\}}}{\Gamma\{\langle{\downarrow}N\rangle\}\{\emptyset\}}$$

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$\rightarrow$ **Weak focusing:**

$$\mathsf{KN} \xrightarrow[\text{simulation}]{} \mathsf{KNwF} + \mathsf{cut}$$

Completeness of focusing

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$\rightarrow$ **Weak focusing:**

$$\mathsf{KN} \longrightarrow \mathsf{KNwF} + \mathsf{cut} \xrightarrow[\text{cut-elimination}]{} \mathsf{KNwF}$$

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Problem: weakening on negative formula!

$\rightarrow$ **Weak focusing:**

$$\mathsf{KN} \longrightarrow \mathsf{KNwF} + \mathsf{cut} \longrightarrow \mathsf{KNwF} \xrightarrow[\text{rules permutation}]{} \mathsf{KNF}$$

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$\rightarrow$ **Synthetic connectives:**

$$\mathsf{KN} \longrightarrow \mathsf{KNF} + \mathsf{cut} \longrightarrow \mathsf{KNF}$$

Completeness of focusing

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$$\mathsf{KN} \longrightarrow \mathsf{KNS} + \mathsf{cut} \longrightarrow \mathsf{KNS}$$

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Focused system KNF**:**

$$\mathsf{dec}\ \frac{\Gamma\{P, \langle P \rangle\}}{\Gamma\{P\}} \qquad \Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \bar{\vee}\ \frac{\Gamma\{A, B\}}{\Gamma\{A \mathbin{\bar{\vee}} B\}} \qquad \bar{\wedge}\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \mathbin{\bar{\wedge}} B\}} \qquad \mathsf{sto}\ \frac{\Gamma\{P\}}{\Gamma\{\uparrow P\}}$$

$$\mathsf{id}\ \frac{}{\Gamma\{\bar{a}, \langle a \rangle\}} \qquad \mathsf{k}^{\diamond}\ \frac{\Gamma\{[\langle A \rangle, \Delta]\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} \qquad \overset{+}{\wedge}\ \frac{\Gamma\{\langle A \rangle\} \quad \Gamma\{\langle B \rangle\}}{\Gamma\{\langle A \mathbin{\overset{+}{\wedge}} B \rangle\}} \qquad \overset{+}{\vee}_i\ \frac{\Gamma\{\langle A_i \rangle\}}{\Gamma\{\langle A_1 \mathbin{\overset{+}{\vee}} A_2 \rangle\}} \qquad \mathsf{rel}\ \frac{\Gamma\{N\}}{\Gamma\{\langle \downarrow N \rangle\}}$$

Focused modal rules:

$$\mathsf{d}^{\diamond}\ \frac{\Gamma\{[\langle A \rangle]\}}{\Gamma\{\langle \diamond A \rangle\}} \qquad \mathsf{t}^{\diamond}\ \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}} \qquad \mathsf{b}^{\diamond}\ \frac{\Gamma\{[\Delta], \langle A \rangle\}}{\Gamma\{[\Delta, \langle \diamond A \rangle]\}} \qquad 4^{\diamond}\ \frac{\Gamma\{[\langle \diamond A \rangle, \Delta]\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} \qquad 5^{\diamond}\ \frac{\Gamma\{\emptyset\}\{\langle \diamond A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}\{\emptyset\}}$$

Focused system KNF**:**

$$\mathsf{dec}\ \frac{\Gamma\{P, \langle P\rangle\}}{\Gamma\{P\}} \qquad \Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \bar{\vee}\ \frac{\Gamma\{A, B\}}{\Gamma\{A \mathbin{\bar{\vee}} B\}} \qquad \bar{\wedge}\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \mathbin{\bar{\wedge}} B\}} \qquad \mathsf{sto}\ \frac{\Gamma\{P\}}{\Gamma\{\uparrow P\}}$$

$$\mathsf{id}\ \frac{}{\Gamma\{\bar{a}, \langle a\rangle\}} \qquad \mathsf{k}^{\Diamond}\ \frac{\Gamma\{[\langle A\rangle, \Delta]\}}{\Gamma\{\langle \Diamond A\rangle, [\Delta]\}} \qquad \overset{+}{\wedge}\ \frac{\Gamma\{\langle A\rangle\} \quad \Gamma\{\langle B\rangle\}}{\Gamma\{\langle A \mathbin{\overset{+}{\wedge}} B\rangle\}} \qquad \overset{+}{\vee}_i\ \frac{\Gamma\{\langle A_i\rangle\}}{\Gamma\{\langle A_1 \mathbin{\overset{+}{\vee}} A_2\rangle\}} \qquad \mathsf{rel}\ \frac{\Gamma\{N\}}{\Gamma\{\langle \downarrow N\rangle\}}$$

Focused modal rules:

$$\mathsf{d}^{\Diamond}\ \frac{\Gamma\{[\langle A\rangle]\}}{\Gamma\{\langle \Diamond A\rangle\}} \qquad \mathsf{t}^{\Diamond}\ \frac{\Gamma\{\langle A\rangle\}}{\Gamma\{\langle \Diamond A\rangle\}} \qquad \mathsf{b}^{\Diamond}\ \frac{\Gamma\{[\Delta], \langle A\rangle\}}{\Gamma\{[\Delta, \langle \Diamond A\rangle]\}} \qquad 4^{\Diamond}\ \frac{\Gamma\{[\langle \Diamond A\rangle, \Delta]\}}{\Gamma\{\langle \Diamond A\rangle, [\Delta]\}} \qquad 5^{\Diamond}\ \frac{\Gamma\{\emptyset\}\{\langle \Diamond A\rangle\}}{\Gamma\{\langle \Diamond A\rangle\}\{\emptyset\}}$$

The synthetic nested system

Synthetic system KNS**:**

$$\mathsf{dec}\ \frac{\Gamma\{P, \langle P\rangle\}}{\Gamma\{P\}} \qquad\qquad \mathsf{neg}\ \frac{\{\Gamma\{\Delta\}\}_{\Delta \preccurlyeq N}}{\Gamma\{N\}}$$

$$\mathsf{id}\ \frac{}{\Gamma\{\bar{a}, \langle a\rangle\}} \quad \mathsf{k}^{\diamond}\ \frac{\Gamma\{[\langle A\rangle, \Delta]\}}{\Gamma\{\langle \diamond A\rangle, [\Delta]\}} \quad \overset{+}{\wedge}\ \frac{\Gamma\{\langle A\rangle\} \quad \Gamma\{\langle B\rangle\}}{\Gamma\{\langle A \overset{+}{\wedge} B\rangle\}} \quad \overset{+}{\vee}_i\ \frac{\Gamma\{\langle A_i\rangle\}}{\Gamma\{\langle A_1 \overset{+}{\vee} A_2\rangle\}} \quad \mathsf{rel}\ \frac{\Gamma\{N\}}{\Gamma\{\langle \downarrow N\rangle\}}$$

Focused modal rules:

$$\mathsf{d}^{\diamond}\ \frac{\Gamma\{[\langle A\rangle]\}}{\Gamma\{\langle \diamond A\rangle\}} \quad \mathsf{t}^{\diamond}\ \frac{\Gamma\{\langle A\rangle\}}{\Gamma\{\langle \diamond A\rangle\}} \quad \mathsf{b}^{\diamond}\ \frac{\Gamma\{[\Delta], \langle A\rangle\}}{\Gamma\{[\Delta, \langle \diamond A\rangle]\}} \quad 4^{\diamond}\ \frac{\Gamma\{[\langle \diamond A\rangle, \Delta]\}}{\Gamma\{\langle \diamond A\rangle, [\Delta]\}} \quad 5^{\diamond}\ \frac{\Gamma\{\emptyset\}\{\langle \diamond A\rangle\}}{\Gamma\{\langle \diamond A\rangle\}\{\emptyset\}}$$

The synthetic nested system

Synthetic system KNS**:**

$$\mathsf{pos}\ \frac{\Delta \preccurlyeq \bar{P} \quad \Gamma\{P, \langle\Delta\rangle\}}{\Gamma\{P\}} \qquad \mathsf{neg}\ \frac{\{\Gamma\{\Delta\}\}_{\Delta \preccurlyeq N}}{\Gamma\{N\}}$$

$$\mathsf{id}\ \frac{}{\Gamma\{\bar{a}, \langle a\rangle\}} \qquad \mathsf{k}^{\diamond}\ \frac{\Gamma\{[\langle A\rangle, \Delta]\}}{\Gamma\{\langle\diamond A\rangle, [\Delta]\}} \qquad \mathsf{split}\ \frac{\Gamma\{\langle\Delta_1\rangle\} \quad \Gamma\{\langle\Delta_2\rangle\}}{\Gamma\{\langle\Delta_1, \Delta_2\rangle\}} \qquad \mathsf{rel}\ \frac{\Gamma\{N\}}{\Gamma\{\langle\downarrow N\rangle\}}$$

Focused modal rules:

$$\mathsf{d}^{\diamond}\ \frac{\Gamma\{[\langle A\rangle]\}}{\Gamma\{\langle\diamond A\rangle\}} \qquad \mathsf{t}^{\diamond}\ \frac{\Gamma\{\langle A\rangle\}}{\Gamma\{\langle\diamond A\rangle\}} \qquad \mathsf{b}^{\diamond}\ \frac{\Gamma\{[\Delta], \langle A\rangle\}}{\Gamma\{[\Delta, \langle\diamond A\rangle]\}} \qquad 4^{\diamond}\ \frac{\Gamma\{[\langle\diamond A\rangle, \Delta]\}}{\Gamma\{\langle\diamond A\rangle, [\Delta]\}} \qquad 5^{\diamond}\ \frac{\Gamma\{\emptyset\}\{\langle\diamond A\rangle\}}{\Gamma\{\langle\diamond A\rangle\}\{\emptyset\}}$$

The synthetic nested system

Synthetic substructure matching:

$$\preccurlyeq\bar{\vee}\ \frac{\Gamma \preccurlyeq M \quad \Delta \preccurlyeq N}{\Gamma, \Delta \preccurlyeq M \mathbin{\bar{\vee}} N} \qquad \preccurlyeq\bar{\wedge}_i\ \frac{\Gamma \preccurlyeq N_i}{\Gamma \preccurlyeq N_1 \mathbin{\bar{\wedge}} N_2} \qquad \preccurlyeq\Box\ \frac{\Gamma \preccurlyeq N}{[\Gamma] \preccurlyeq \Box N} \qquad \preccurlyeq\uparrow\ \frac{}{P \preccurlyeq \uparrow P} \qquad \preccurlyeq\mathsf{id}\ \frac{}{\bar{a} \preccurlyeq \bar{a}}$$

Synthetic system KNS**:**

$$\mathsf{pos}\ \frac{\Delta \preccurlyeq \bar{P} \quad \Gamma\{P, \langle\Delta\rangle\}}{\Gamma\{P\}} \qquad \mathsf{neg}\ \frac{\big\{\Gamma\{\Delta\}\big\}_{\Delta \preccurlyeq N}}{\Gamma\{N\}}$$

$$\mathsf{id}\ \frac{}{\Gamma\{\bar{a}, \langle a\rangle\}} \qquad \mathsf{k}^{\diamond}\ \frac{\Gamma\{[\langle A\rangle, \Delta]\}}{\Gamma\{\langle\Diamond A\rangle, [\Delta]\}} \qquad \mathsf{split}\ \frac{\Gamma\{\langle\Delta_1\rangle\} \quad \Gamma\{\langle\Delta_2\rangle\}}{\Gamma\{\langle\Delta_1, \Delta_2\rangle\}} \qquad \mathsf{rel}\ \frac{\Gamma\{N\}}{\Gamma\{\langle\downarrow N\rangle\}}$$

Focused modal rules:

$$\mathsf{d}^{\diamond}\ \frac{\Gamma\{[\langle A\rangle]\}}{\Gamma\{\langle\Diamond A\rangle\}} \qquad \mathsf{t}^{\diamond}\ \frac{\Gamma\{\langle A\rangle\}}{\Gamma\{\langle\Diamond A\rangle\}} \qquad \mathsf{b}^{\diamond}\ \frac{\Gamma\{[\Delta], \langle A\rangle\}}{\Gamma\{[\Delta, \langle\Diamond A\rangle]\}} \qquad \mathsf{4}^{\diamond}\ \frac{\Gamma\{[\langle\Diamond A\rangle, \Delta]\}}{\Gamma\{\langle\Diamond A\rangle, [\Delta]\}} \qquad \mathsf{5}^{\diamond}\ \frac{\Gamma\{\emptyset\}\{\langle\Diamond A\rangle\}}{\Gamma\{\langle\Diamond A\rangle\}\{\emptyset\}}$$

Synthetic substructure matching:

$$\preccurlyeq\bar{\vee}\ \frac{\Gamma \preccurlyeq M \quad \Delta \preccurlyeq N}{\Gamma, \Delta \preccurlyeq M \mathbin{\bar{\vee}} N} \qquad \preccurlyeq\bar{\wedge}_i\ \frac{\Gamma \preccurlyeq N_i}{\Gamma \preccurlyeq N_1 \mathbin{\bar{\wedge}} N_2} \qquad \preccurlyeq\Box\ \frac{\Gamma \preccurlyeq N}{[\Gamma] \preccurlyeq \Box N} \qquad \preccurlyeq\uparrow\ \frac{}{P \preccurlyeq \uparrow P} \qquad \preccurlyeq\mathrm{id}\ \frac{}{\bar{a} \preccurlyeq \bar{a}}$$

Synthetic system KNS**:**

$$\mathrm{pos}\ \frac{\Delta \preccurlyeq \bar{P} \quad \Gamma\{P, \langle\Delta\rangle\}}{\Gamma\{P\}} \qquad \mathrm{neg}\ \frac{\left\{\Gamma\{\Delta\}\right\}_{\Delta \preccurlyeq N}}{\Gamma\{N\}}$$

$$\mathrm{id}\ \frac{}{\Gamma\{\bar{a}, \langle\bar{a}\rangle\}} \qquad \mathrm{k}^{\diamond}\ \frac{\Gamma\{[\langle A\rangle, \Delta]\}}{\Gamma\{\langle\Diamond A\rangle, [\Delta]\}} \qquad \mathrm{split}\ \frac{\Gamma\{\langle\Delta_1\rangle\} \quad \Gamma\{\langle\Delta_2\rangle\}}{\Gamma\{\langle\Delta_1, \Delta_2\rangle\}} \qquad \mathrm{rel}\ \frac{\Gamma\{\bar{P}\}}{\Gamma\{\langle P\rangle\}}$$

Focused modal rules:

$$\mathrm{d}^{\diamond}\ \frac{\Gamma\{[\langle A\rangle]\}}{\Gamma\{\langle\Diamond A\rangle\}} \qquad \mathrm{t}^{\diamond}\ \frac{\Gamma\{\langle A\rangle\}}{\Gamma\{\langle\Diamond A\rangle\}} \qquad \mathrm{b}^{\diamond}\ \frac{\Gamma\{[\Delta], \langle A\rangle\}}{\Gamma\{[\Delta, \langle\Diamond A\rangle]\}} \qquad 4^{\diamond}\ \frac{\Gamma\{[\langle\Diamond A\rangle, \Delta]\}}{\Gamma\{\langle\Diamond A\rangle, [\Delta]\}} \qquad 5^{\diamond}\ \frac{\Gamma\{\emptyset\}\{\langle\Diamond A\rangle\}}{\Gamma\{\langle\Diamond A\rangle\}\{\emptyset\}}$$

The synthetic nested system

Synthetic substructure matching:

$$\preccurlyeq\bar{\vee}\ \frac{\Gamma \preccurlyeq M \quad \Delta \preccurlyeq N}{\Gamma, \Delta \preccurlyeq M \mathbin{\bar{\vee}} N} \qquad \preccurlyeq\bar{\wedge}_i\ \frac{\Gamma \preccurlyeq N_i}{\Gamma \preccurlyeq N_1 \mathbin{\bar{\wedge}} N_2} \qquad \preccurlyeq\Box\ \frac{\Gamma \preccurlyeq N}{[\Gamma] \preccurlyeq \Box N} \qquad \preccurlyeq\uparrow\ \frac{}{P \preccurlyeq \uparrow P} \qquad \preccurlyeq\mathsf{id}\ \frac{}{\bar{a} \preccurlyeq \bar{a}}$$

Synthetic system KNS**:**

$$\mathsf{pos}\ \frac{\Delta \preccurlyeq \bar{P} \quad \Gamma\{P, \langle\Delta\rangle\}}{\Gamma\{P\}} \qquad \mathsf{neg}\ \frac{\{\Gamma\{\Delta\}\}_{\Delta \preccurlyeq N}}{\Gamma\{N\}}$$

$$\mathsf{id}\ \frac{}{\Gamma\{\bar{a}, \langle\bar{a}\rangle\}} \qquad \mathsf{k}^{\langle\rangle}\ \frac{\Gamma\{[\langle\Delta\rangle, \Omega]\}}{\Gamma\{\langle[\Delta]\rangle, [\Omega]\}} \qquad \mathsf{split}\ \frac{\Gamma\{\langle\Delta_1\rangle\} \quad \Gamma\{\langle\Delta_2\rangle\}}{\Gamma\{\langle\Delta_1, \Delta_2\rangle\}} \qquad \mathsf{rel}\ \frac{\Gamma\{\bar{P}\}}{\Gamma\{\langle P\rangle\}}$$

Focused modal rules:

$$\mathsf{d}^{\diamond}\ \frac{\Gamma\{[\langle A\rangle]\}}{\Gamma\{\langle\Diamond A\rangle\}} \qquad \mathsf{t}^{\diamond}\ \frac{\Gamma\{\langle A\rangle\}}{\Gamma\{\langle\Diamond A\rangle\}} \qquad \mathsf{b}^{\diamond}\ \frac{\Gamma\{[\Delta], \langle A\rangle\}}{\Gamma\{[\Delta, \langle\Diamond A\rangle]\}} \qquad 4^{\diamond}\ \frac{\Gamma\{[\langle\Diamond A\rangle, \Delta]\}}{\Gamma\{\langle\Diamond A\rangle, [\Delta]\}} \qquad 5^{\diamond}\ \frac{\Gamma\{\emptyset\}\{\langle\Diamond A\rangle\}}{\Gamma\{\langle\Diamond A\rangle\}\{\emptyset\}}$$

The synthetic nested system

Synthetic substructure matching:

$$\preccurlyeq\bar{\vee}\ \frac{\Gamma \preccurlyeq M \quad \Delta \preccurlyeq N}{\Gamma, \Delta \preccurlyeq M \mathbin{\bar{\vee}} N} \qquad \preccurlyeq\bar{\wedge}_i\ \frac{\Gamma \preccurlyeq N_i}{\Gamma \preccurlyeq N_1 \mathbin{\bar{\wedge}} N_2} \qquad \preccurlyeq\Box\ \frac{\Gamma \preccurlyeq N}{[\Gamma] \preccurlyeq \Box N} \qquad \preccurlyeq\uparrow\ \frac{}{P \preccurlyeq \uparrow P} \qquad \preccurlyeq\mathsf{id}\ \frac{}{\bar{a} \preccurlyeq \bar{a}}$$

Synthetic system KNS**:**

$$\mathsf{pos}\ \frac{\Delta \preccurlyeq \bar{P} \quad \Gamma\{P, \langle\Delta\rangle\}}{\Gamma\{P\}} \qquad \mathsf{neg}\ \frac{\left\{\Gamma\{\Delta\}\right\}_{\Delta \preccurlyeq N}}{\Gamma\{N\}}$$

$$\mathsf{id}\ \frac{}{\Gamma\{\bar{a}, \langle\bar{a}\rangle\}} \qquad \mathsf{k}^{\langle\rangle}\ \frac{\Gamma\{[\langle\Delta\rangle, \Omega]\}}{\Gamma\{\langle[\Delta]\rangle, [\Omega]\}} \qquad \mathsf{split}\ \frac{\Gamma\{\langle\Delta_1\rangle\} \quad \Gamma\{\langle\Delta_2\rangle\}}{\Gamma\{\langle\Delta_1, \Delta_2\rangle\}} \qquad \mathsf{rel}\ \frac{\Gamma\{\bar{P}\}}{\Gamma\{\langle P\rangle\}}$$

Synthetic modal rules:

$$\mathsf{d}^{\langle\rangle}\ \frac{\Gamma\{[\langle\Delta\rangle]\}}{\Gamma\{\langle[\Delta]\rangle\}} \qquad \mathsf{t}^{\langle\rangle}\ \frac{\Gamma\{\langle\Delta\rangle\}}{\Gamma\{\langle[\Delta]\rangle\}} \qquad \mathsf{b}^{\langle\rangle}\ \frac{\Gamma\{[\Omega], \langle\Delta\rangle\}}{\Gamma\{[\Omega, \langle[\Delta]\rangle]\}} \qquad 4^{\langle\rangle}\ \frac{\Gamma\{[\Omega, \langle[\Delta]\rangle]\}}{\Gamma\{[\Omega], \langle[\Delta]\rangle\}} \qquad 5^{\langle\rangle}\ \frac{\Gamma\{\langle[\Delta]\rangle\}\{\emptyset\}}{\Gamma\{\emptyset\}\{\langle[\Delta]\rangle\}}$$

In action...

Synthetic connectives: $\text{neg}\ \dfrac{\{\Gamma\{\Delta\}\}_{\Delta \preccurlyeq N}}{\Gamma\{N\}}$ and $\text{pos}\ \dfrac{\Delta \preccurlyeq \bar{P} \quad \Gamma\{P, \langle\Delta\rangle\}}{\Gamma\{P\}}$

Structural modal rules : distinct modal phase and action on substructures

$$
\text{neg}\ \dfrac{
\text{pos}\ \dfrac{\mathsf{k}^{\langle\rangle}\ \dfrac{\text{id}\ \dfrac{}{\Diamond(a \overset{+}{\vee} b), [\langle \bar{a} \rangle, \bar{a}]}}{\Diamond(a \overset{+}{\vee} b), \langle \bar{a} \rangle, [\bar{a}]}}{\Diamond(a \overset{+}{\vee} b), [\bar{a}]}
\qquad
\text{pos}\ \dfrac{\mathsf{k}^{\langle\rangle}\ \dfrac{\text{id}\ \dfrac{}{\Diamond(a \overset{+}{\vee} b), [\langle \bar{b} \rangle, \bar{\ }]}}{\Diamond(a \overset{+}{\vee} b), \langle \bar{b} \rangle, [\bar{b}]}}{\Diamond(a \overset{+}{\vee} b), [\bar{b}]}
}{\Diamond(a \overset{+}{\vee} b), \Box(\bar{a} \overset{-}{\wedge} \bar{b})}
$$

$$
{\preccurlyeq}\Box\ \dfrac{{\preccurlyeq}\overset{-}{\wedge}\ \dfrac{{\preccurlyeq}\text{id}\ \dfrac{}{\bar{a} \preccurlyeq \bar{a}}}{\bar{a} \preccurlyeq \bar{a} \overset{-}{\wedge} \bar{b}}}{[\bar{a}] \preccurlyeq \Box(\bar{a} \overset{-}{\wedge} \bar{b})}
\qquad\qquad
{\preccurlyeq}\Box\ \dfrac{{\preccurlyeq}\overset{-}{\wedge}\ \dfrac{{\preccurlyeq}\text{id}\ \dfrac{}{\bar{b} \preccurlyeq \bar{a}}}{\bar{b} \preccurlyeq \bar{a} \overset{-}{\wedge} \bar{b}}}{[\bar{b}] \preccurlyeq \Box(\bar{a} \overset{-}{\wedge} \bar{b})}
$$

Synthetic permutation

$$
\mathsf{cut}\;\dfrac{\mathsf{rel}\;\dfrac{\overset{\mathcal{D}_1}{\Gamma\{N\}\{P\}}}{\Gamma\{{\color{red}\langle}\downarrow N{\color{red}\rangle}\}\{P\}} \qquad \overset{\mathcal{D}_2}{\Gamma\{\emptyset\}\{\bar{P}\}}}{\Gamma\{{\color{red}\langle}\downarrow N{\color{red}\rangle}\}\{\emptyset\}}
\quad\rightsquigarrow\quad
\mathsf{rel}\;\dfrac{\mathsf{cut}\;\dfrac{\overset{\mathcal{D}_1}{\Gamma\{N\}\{P\}} \qquad \mathsf{weak}\;\dfrac{\overset{\mathcal{D}_2}{\Gamma\{\emptyset\}\{\bar{P}\}}}{\Gamma\{N\}\{\bar{P}\}}}{\Gamma\{N\}\{\emptyset\}}}{\Gamma\{{\color{red}\langle}\downarrow N{\color{red}\rangle}\}\{\emptyset\}}
$$

Synthetic permutation

$$\text{cut}\;\dfrac{\text{rel}\;\dfrac{\text{neg}\;\dfrac{\left\{\begin{array}{c}\mathcal{D}_\Delta\\ \Gamma\{\Delta\}\{P\}\end{array}\right\}_{\Delta\preccurlyeq\bar{Q}}}{\Gamma\{\bar{Q}\}\{P\}}}{\Gamma\{\langle Q\rangle\}\{P\}} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma\{\emptyset\}\{\bar{P}\}\end{array}}{\Gamma\{\langle Q\rangle\}\{\emptyset\}} \quad\rightsquigarrow\quad \text{rel}\;\dfrac{\text{neg}\;\dfrac{\left\{\text{cut}\;\dfrac{\begin{array}{c}\mathcal{D}_\Delta\\ \Gamma\{\Delta\}\{P\}\end{array}\qquad \text{weak}\;\dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma\{\emptyset\}\{\bar{P}\}\end{array}}{\Gamma\{\Delta\}\{\bar{P}\}}}{\Gamma\{\Delta\}\{\emptyset\}}\right\}_{\Delta\preccurlyeq\bar{Q}}}{\Gamma\{\bar{Q}\}\{\emptyset\}}}{\Gamma\{\langle Q\rangle\}\{\emptyset\}}$$

Synthetic permutation

$$
\mathsf{cut}\ \dfrac{\mathsf{rel}\ \dfrac{\mathsf{neg}\ \dfrac{\left\{\begin{matrix}\mathcal{D}_\Delta \\ \vdots \\ \Gamma\{\Delta\}\{P\}\end{matrix}\right\}_{\Delta \preccurlyeq \bar{Q}}}{\Gamma\{\bar{Q}\}\{P\}}}{\Gamma\{\langle Q\rangle\}\{P\}} \qquad \begin{matrix}\mathcal{D}_2 \\ \vdots \\ \Gamma\{\emptyset\}\{\bar{P}\}\end{matrix}}{\Gamma\{\langle Q\rangle\}\{\emptyset\}}
\quad \rightsquigarrow \quad
\mathsf{rel}\ \dfrac{\mathsf{neg}\ \dfrac{\left\{\mathsf{cut}\ \dfrac{\begin{matrix}\mathcal{D}_\Delta \\ \vdots \\ \Gamma\{\Delta\}\{P\}\end{matrix} \qquad \mathsf{weak}\ \dfrac{\begin{matrix}\mathcal{D}_2 \\ \vdots \\ \Gamma\{\emptyset\}\{\bar{P}\}\end{matrix}}{\Gamma\{\Delta\}\{\bar{P}\}}}{\Gamma\{\Delta\}\{\emptyset\}}\right\}_{\Delta \preccurlyeq \bar{Q}}}{\Gamma\{\bar{Q}\}\{\emptyset\}}}{\Gamma\{\langle Q\rangle\}\{\emptyset\}}
$$

$$
\mathsf{KN} \longrightarrow \mathsf{KNS} + \mathsf{cut} \longrightarrow \mathsf{KNS}
$$

Conclusion and perspectives

- Focused and synthetic variants of nested systems for the S5-cube
- Internal proof of focusing via cut-elimination

Conclusion and perspectives

- Focused and synthetic variants of nested systems for the S5-cube
- Internal proof of focusing via cut-elimination

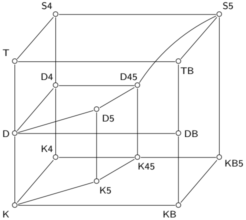
- Intuitionistic modal logics : IKN $\rightarrow$ IKNF?
- Other proof formalisms: hypersequents...
- Exponentials in linear logic

Cut-elimination

Theorem Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$ be 45-closed.
If a sequent Γ is provable in $\mathsf{KNF} + \mathsf{X}^{\diamond} + \mathsf{Cut}$, then it is also provable in $\mathsf{KNF} + \mathsf{X}^{\diamond}$.

$$\mathsf{Cut} = \left\{ \mathsf{cut}_1 \ \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}; \mathsf{cut}_2 \ \frac{\Gamma\{\langle P \rangle\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}; \mathsf{cut}_3 \ \frac{\Gamma\{\langle Q \rangle\}\{P\} \quad \Gamma\{\emptyset\}\{\bar{P}\}}{\Gamma\{\langle Q \rangle\}\{\emptyset\}} \right\}$$

$$clo(\mathsf{X}) = \begin{cases} \mathsf{X} \cup \{4\} & \text{if } \{\mathsf{b}, 5\} \subseteq \mathsf{X} \text{ or if } \{\mathsf{t}, 5\} \subseteq \mathsf{X} \\ \mathsf{X} \cup \{5\} & \text{if } \{\mathsf{b}, 4\} \subseteq \mathsf{X} \\ \mathsf{X} & \text{otherwise} \end{cases}$$



Cut-elimination proof

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{}
\LeftLabel{id}
\UnaryInfC{$[\bar{a}, a], [\Diamond a]$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$[\bar{a}], \Diamond a, [\Diamond a]$}
\LeftLabel{$\Box$}
\UnaryInfC{$\Box\bar{a}, \Diamond a, [\Diamond a]$}
\LeftLabel{$\mathsf{b}^{\Diamond}$}
\UnaryInfC{$\Box\bar{a}, [\Diamond\Diamond a, \Diamond a]$}
\AxiomC{}
\LeftLabel{id}
\UnaryInfC{$\Box\bar{a}, [[[\bar{a}, a]]]$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$\Box\bar{a}, [[[\bar{a}], \Diamond a]]$}
\LeftLabel{$\Box$}
\UnaryInfC{$\Box\bar{a}, [[\Box\bar{a}, \Diamond a]]$}
\LeftLabel{$4^{\Diamond}$}
\UnaryInfC{$\Box\bar{a}, [[\Box\bar{a}], \Diamond a]$}
\LeftLabel{$\Box$}
\UnaryInfC{$\Box\bar{a}, [\Box\Box\bar{a}, \Diamond a]$}
\LeftLabel{cut}
\BinaryInfC{$\Box\bar{a}, [\Diamond a]$}
\end{prooftree}
\quad \rightsquigarrow \quad
\begin{prooftree}
\AxiomC{}
\LeftLabel{id}
\UnaryInfC{$[\bar{a}, a], [\;]$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$[\bar{a}], \Diamond a, [\;]$}
\LeftLabel{$\Box$}
\UnaryInfC{$\Box\bar{a}, \Diamond a, [\;]$}
\LeftLabel{$5^{\Diamond}$}
\UnaryInfC{$\Box\bar{a}, [\Diamond a]$}
\end{prooftree}
$$

Completeness proof

KN

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{[\bar{a}, a], [\ ]}\ \mathsf{id}}{[\bar{a}], \Diamond a, [\ ]}\ \Diamond}{\Box\bar{a}, \Diamond a, [\ ]}\ \Box}{\Box\bar{a}, [\Diamond a]}\ 5^{\Diamond}}{\Box\bar{a}, \Box\Diamond a}\ \Box
$$

Completeness proof

simulation

$$\mathsf{KN} \longrightarrow \mathsf{KNwF} + \mathsf{cut}$$

$$\Box\;\dfrac{5^{\Diamond}\;\dfrac{\Box\;\dfrac{\Diamond\;\dfrac{\mathsf{id}\;\dfrac{}{[\bar{a}, a], [\ ]}}{[\bar{a}], \Diamond a, [\ ]}}{\Box\bar{a}, \Diamond a, [\ ]}}{\Box\bar{a}, [\Diamond a]}}{\Box\bar{a}, \Box\Diamond a}$$

$$\Box\;\dfrac{\mathsf{sto}\;\dfrac{\mathsf{cut}_1\;\dfrac{\Box\;\dfrac{\mathsf{cut}_1\;\dfrac{\mathsf{dec}\;\dfrac{\mathsf{id}\;\dfrac{}{[\bar{a}, \langle a\rangle], \Diamond a, [\Diamond a]}}{[\bar{a}, a], \Diamond a, [\Diamond a]} \qquad \mathsf{dec}\;\dfrac{\Diamond\;\dfrac{\mathsf{id}\;\dfrac{}{[\bar{a}, \bar{a}, \langle a\rangle], \Diamond a, [\Diamond a]}}{[\bar{a}, \bar{a}], \Diamond a, \langle\Diamond a\rangle, [\Diamond a]}}{[\bar{a}, \bar{a}], \Diamond a, [\Diamond a]}}{[\bar{a}], \Diamond a, [\Diamond a]}}{\Box\bar{a}, \Diamond a, [\Diamond a]} \qquad \mathsf{dec}\;\dfrac{5^{\Diamond}\;\dfrac{\Box\;\dfrac{\Diamond\;\dfrac{\mathsf{id}\;\dfrac{}{\Box\bar{a}, [\bar{a}, \langle a\rangle], [\Diamond a]}}{\Box\bar{a}, [\bar{a}], \langle\Diamond a\rangle, [\Diamond a]}}{\Box\bar{a}, \Box\bar{a}, \langle\Diamond a\rangle, [\Diamond a]}}{\Box\bar{a}, \Box\bar{a}, [\Diamond a, \langle\Diamond a\rangle]}}{\Box\bar{a}, \Box\bar{a}, [\Diamond a]}}{\Box\bar{a}, [\Diamond a]}}{\Box\bar{a}, [\uparrow\Diamond a]}}{\Box\bar{a}, \Box\uparrow\Diamond a}$$

$$\mathsf{KN} \longrightarrow \mathsf{KNwF} + \mathsf{cut} \xrightarrow{\text{cut-elimination}} \mathsf{KNwF}$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{}\LeftLabel{id}
\UnaryInfC{$[\bar{a}, \langle a \rangle], \Diamond a, [\Diamond a]$}
\LeftLabel{dec}
\UnaryInfC{$[\bar{a}, a], \Diamond a, [\Diamond a]$}
\AxiomC{}
\LeftLabel{id}
\UnaryInfC{$[\bar{a}, \bar{a}, \langle a \rangle], \Diamond a, [\Diamond a]$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$[\bar{a}, \bar{a}], \Diamond a, \langle \Diamond a \rangle, [\Diamond a]$}
\LeftLabel{dec}
\UnaryInfC{$[\bar{a}, \bar{a}], \Diamond a, [\Diamond a]$}
\LeftLabel{cut_1}
\BinaryInfC{$[\bar{a}], \Diamond a, [\Diamond a]$}
\LeftLabel{$\Box$}
\UnaryInfC{$\Box \bar{a}, \Diamond a, [\Diamond a]$}
\AxiomC{}
\LeftLabel{id}
\UnaryInfC{$\Box \bar{a}, [\bar{a}, \langle a \rangle], [\Diamond a]$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$\Box \bar{a}, [\bar{a}], \langle \Diamond a \rangle, [\Diamond a]$}
\LeftLabel{$\Box$}
\UnaryInfC{$\Box \bar{a}, \Box \bar{a}, \langle \Diamond a \rangle, [\Diamond a]$}
\LeftLabel{$5^{\Diamond}$}
\UnaryInfC{$\Box \bar{a}, \Box \bar{a}, [\Diamond a, \langle \Diamond a \rangle]$}
\LeftLabel{dec}
\UnaryInfC{$\Box \bar{a}, \Box \bar{a}, [\Diamond a]$}
\LeftLabel{cut_1}
\BinaryInfC{$\Box \bar{a}, [\Diamond a]$}
\LeftLabel{sto}
\UnaryInfC{$\Box \bar{a}, [\uparrow \Diamond a]$}
\LeftLabel{$\Box$}
\UnaryInfC{$\Box \bar{a}, \Box \uparrow \Diamond a$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{}
\LeftLabel{id}
\UnaryInfC{$[\bar{a}, \langle a \rangle], [\Diamond a]$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$[\bar{a}], \langle \Diamond a \rangle, [\Diamond a]$}
\LeftLabel{$\Box$}
\UnaryInfC{$\Box \bar{a}, \langle \Diamond a \rangle, [\Diamond a]$}
\LeftLabel{$5^{\Diamond}$}
\UnaryInfC{$\Box \bar{a}, [\Diamond a, \langle \Diamond a \rangle]$}
\LeftLabel{dec}
\UnaryInfC{$\Box \bar{a}, [\Diamond a]$}
\LeftLabel{sto}
\UnaryInfC{$\Box \bar{a}, [\uparrow \Diamond a]$}
\LeftLabel{$\Box$}
\UnaryInfC{$\Box \bar{a}, \Box \uparrow \Diamond a$}
\end{prooftree}
$$

Completeness proof

rules permutation

$$\mathsf{KN} \longrightarrow \mathsf{KNwF} + \mathsf{cut} \longrightarrow \mathsf{KNwF} \longrightarrow \mathsf{KNF}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[\bar{a}, \langle a \rangle], [\Diamond a]}\,\mathsf{id}}{[\bar{a}], \langle \Diamond a \rangle, [\Diamond a]}\,\Diamond}{\Box \bar{a}, \langle \Diamond a \rangle, [\Diamond a]}\,\Box}{\Box \bar{a}, [\Diamond a, \langle \Diamond a \rangle]}\,5^{\Diamond}}{\dfrac{\Box \bar{a}, [\Diamond a]}{\Box \bar{a}, [\uparrow \Diamond a]}\,\mathsf{sto}}\,\mathsf{dec}}{\Box \bar{a}, \Box \uparrow \Diamond a}\,\Box$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[\bar{a}, \langle a \rangle], [\Diamond a]}\,\mathsf{id}}{[\bar{a}], \langle \Diamond a \rangle, [\Diamond a]}\,\Diamond}{[\bar{a}], [\Diamond a, \langle \Diamond a \rangle]}\,5^{\Diamond}}{[\bar{a}], [\Diamond a]}\,\mathsf{dec}}{\dfrac{[\bar{a}], [\uparrow \Diamond a]}{\Box \bar{a}, [\uparrow \Diamond a]}\,\Box}\,\mathsf{sto}}{\Box \bar{a}, \Box \uparrow \Diamond a}\,\Box$$

In action...

- neg and pos : synthetic connectives
- structural modal rules : modal phase / action on substructures

$$
\mathsf{pos}^{\langle\rangle}\,\dfrac{\mathsf{k}^{\langle\rangle}\,\dfrac{\mathsf{rel}^{\langle\rangle}\,\dfrac{\mathsf{neg}^{\langle\rangle}\,\dfrac{\mathsf{pos}^{\langle\rangle}\,\dfrac{\mathsf{k}^{\langle\rangle}\,\dfrac{\mathsf{id}^{\langle\rangle}\,\dfrac{}{\Diamond{\downarrow}(\bar{a} \mathbin{\bar{\wedge}} {\uparrow} b), \Diamond a, [\langle \bar{a} \rangle, \bar{a}, \bar{b}]}}{\Diamond{\downarrow}(\bar{a} \mathbin{\bar{\wedge}} {\uparrow} b), \Diamond a, \langle [\bar{a}] \rangle, [\bar{a}, \bar{b}]}}{\Diamond{\downarrow}(\bar{a} \mathbin{\bar{\wedge}} {\uparrow} b), \Diamond a, [\bar{a}, \bar{b}]} \qquad \mathsf{pos}^{\langle\rangle}\,\dfrac{\mathsf{id}^{\langle\rangle}\,\dfrac{}{\Diamond{\downarrow}(\bar{a} \mathbin{\bar{\wedge}} {\uparrow} b), \Diamond a, [b, \langle \bar{b} \rangle, \bar{b}]}}{\Diamond{\downarrow}(\bar{a} \mathbin{\bar{\wedge}} {\uparrow} b), \Diamond a, [b, \bar{b}]}}{\Diamond{\downarrow}(\bar{a} \mathbin{\bar{\wedge}} {\uparrow} b), \Diamond a, [\bar{a} \mathbin{\bar{\wedge}} {\uparrow} b, \bar{b}]}}{\Diamond{\downarrow}(\bar{a} \mathbin{\bar{\wedge}} {\uparrow} b), \Diamond a, [\langle a \mathbin{\overset{+}{\vee}} {\downarrow} \bar{b} \rangle, \bar{b}]}}{\Diamond{\downarrow}(\bar{a} \mathbin{\bar{\wedge}} {\uparrow} b), \langle [a \mathbin{\overset{+}{\vee}} {\downarrow} b] \rangle, \Diamond a, [\bar{b}]}}{\Diamond{\downarrow}(\bar{a} \mathbin{\bar{\wedge}} {\uparrow} b), \Diamond a, [\bar{b}]}
$$

$$
\mathsf{pos}^{\langle\rangle}\,\dfrac{\mathsf{k}^{\langle\rangle}\,\dfrac{\mathsf{rel}^{\langle\rangle}\,\dfrac{\mathsf{pos}^{\langle\rangle}\,\dfrac{\mathsf{k}^{\langle\rangle}\,\dfrac{\mathsf{split}^{\langle\rangle}\,\dfrac{\mathsf{id}^{\langle\rangle}\,\dfrac{}{\Diamond(a \mathbin{\overset{+}{\wedge}} b), \Diamond{\downarrow}\bar{a}, [\langle \bar{a} \rangle, \bar{a}, \bar{b}]} \qquad \mathsf{id}^{\langle\rangle}\,\dfrac{}{\Diamond(a \mathbin{\overset{+}{\wedge}} b), \Diamond{\downarrow}\bar{a}, [\langle \bar{b} \rangle, \bar{a}, \bar{b}]}}{\Diamond(a \mathbin{\overset{+}{\wedge}} b), \Diamond{\downarrow}\bar{a}, [\langle \bar{a}, \bar{b} \rangle, \bar{a}, \bar{b}]}}{\Diamond(a \mathbin{\overset{+}{\wedge}} b), \langle [\bar{a}, \bar{b}] \rangle, \Diamond{\downarrow}\bar{a}, [\bar{a}, \bar{b}]}}{\Diamond(a \mathbin{\overset{+}{\wedge}} b), \Diamond{\downarrow}\bar{a}, [\bar{a}, \bar{b}]}}{\Diamond(a \mathbin{\overset{+}{\wedge}} b), \Diamond{\downarrow}\bar{a}, [\langle a \rangle, \bar{b}]}}{\Diamond(a \mathbin{\overset{+}{\wedge}} b), \Diamond{\downarrow}\bar{a}, \langle [a] \rangle, [\bar{b}]}}{\Diamond(a \mathbin{\overset{+}{\wedge}} b), \Diamond{\downarrow}\bar{a}, [\bar{b}]}
$$