Decomposing labelled proof theory for intuitionistic modal logic

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Intuitionistic modal logics

Formulas: $A ::= a \mid A \wedge A \mid A \vee A \mid \bot \mid A \supset A$

Logic IK**:** Intuitionistic Propositional Logic

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(Plotkin and Sterling 1986)

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\mathsf{k3}: & \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) \\
\mathsf{k4}: & (\Diamond A \supset \Box B) \supset \Box(A \supset B) \\
\mathsf{k5}: & \Diamond\bot \supset \bot
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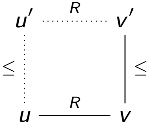
(Plotkin and Sterling 1986)

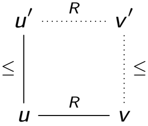
Kripke semantics: (Bi)relational structures $(W, R, \leq)$ (Fischer-Servi 1984)

- a non-empty set W of *worlds*;
- a binary relation $R \subseteq W \times W$;
- and a preorder $\leq$ on W, such that:

(F_1)



(F_2)



Intuitionistic modal logics

Sequent system:

$$\mathsf{k}_{\Box}\ \frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A} \qquad \mathsf{k}_{\Diamond}\ \frac{\Gamma, A \Rightarrow B}{\Box\Gamma, \Diamond A \Rightarrow \Diamond B}$$

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Problem? k3, k4 and k5 are not derivable.

- not a problem for modal type theory...

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Labelled sequent system: (Simpson 1994)

$$\Box_L\ \frac{xRy, \Gamma, x : \Box A, y : A \Rightarrow z : B}{xRy, \Gamma, x : \Box A \Rightarrow z : B} \qquad \Box_R\ \frac{xRy, \Gamma \Rightarrow y : A}{\Gamma \Rightarrow x : \Box A}\ y \text{ is fresh}$$

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Scott-Lemmon axioms: for h, i, j, k natural numbers,

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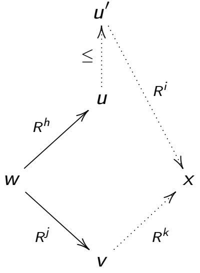
Theorem: (Plotkin and Sterling 1986)
Intuitionistic modal frame $(W, R, \leq)$ validates $\mathsf{g}_{\mathsf{hijk}}$ iff the frame satisfies:

if $wR^h u$ and $wR^j v$ then there exists u' s.t. $u \leq u'$
and there exist x s.t. $u' R^i x$ and $vR^k x$



Extensions - Labelled system

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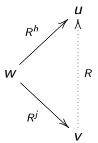
For the special case: $\mathsf{p_{hj}} : (\Diamond^h \Box A \supset \Box^j A) \wedge (\Diamond^j A \supset \Box^h \Diamond A)$

Corresponding rule: (Simpson 1994)

$$\boxtimes_{\mathsf{p_{hj}}} \; \frac{wR^h v, wR^j u, vRu, \Gamma \Rightarrow z : C}{wR^h v, wR^j u, \Gamma \Rightarrow z : C}$$

Theorem:
$\mathsf{labIK} + \boxtimes_{\mathsf{p_{hj}}}$ sound and complete for $\mathsf{IK} + \mathsf{p_{hj}}$ and for frames satisfying



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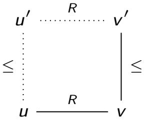
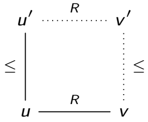
$$\mathsf{F}_1 \frac{xRy, y \leq z, x \leq u, uRz, \Gamma \Rightarrow \Delta}{xRy, y \leq z, \Gamma \Rightarrow \Delta} \; u \text{ fresh} \qquad \mathsf{F}_2 \frac{xRy, x \leq z, y \leq u, zRu, \Gamma \Rightarrow \Delta}{xRy, x \leq z, \Gamma \Rightarrow \Delta} \; u \text{ fresh}$$





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Corresponding rule:

$$\boxtimes_{\mathsf{g_{hijk}}} \frac{wR^h v, wR^j u, u \leq u', u' R^i x, vR^k x, \Gamma \Rightarrow \Delta}{wR^h v, wR^j u, \Gamma \Rightarrow \Delta} \; u', x \text{ fresh}$$

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Results: Soundness, completeness, cut-elimination.