Fully structured proof theory for intuitionistic modal logics

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Abstract

We present a labelled sequent system and a nested sequent system for intuitionistic modal logics equipped with two relation symbols, one for the accessibility relation associated with the Kripke semantics for modal logics and one for the preorder relation associated with the Kripke semantics for intuitionistic logic. Both systems are in close correspondence with the bi-relational Kripke semantics for intuitionistic modal logic.

Keywords: Nested sequents, Labelled sequents, Intuitionistic modal logic, Proof theory.

1 Introduction

Structural proof theoretic accounts of modal logic can adopt the paradigm of *labelled deduction*, in the form of e.g. labelled sequent systems [12,7], or the one of *unlabelled deduction*, in the form of e.g. nested sequent systems [1,9].

These generalisations of the sequent framework, inspired by relational semantics, are needed to treat modalities uniformly. By extending the ordinary sequent structure with *one* extra element, either relational atoms between labels or nested bracketing, they encode respectively graphs or trees in the sequents, giving them enough power to represent modalities.

Similarly, proof systems have been designed for *intuitionistic* modal logic both as labelled [10] and as nested [11,4,3] sequent systems. Surprisingly, in nested and labelled sequents, extending the sequent structure with the same *one* extra element is enough to obtain sound and complete systems.

This no longer matches the relational semantics of these logics, which requires to combine *both* the relation for intuitionistic propositional logic and the one for modal logic. More importantly, it leads to deductive systems that are not entirely satisfactory; they cannot as modularly capture axiomatic extensions (or equivalently, restricted semantical conditions) and, in particular, can only provide decision procedures for a handful of them [10].

This lead us to develop a *fully structured* approach to intuitionistic modal proof theory capturing *both* the modal accessibility relation and the intuitionistic preorder relation. A *fully labelled* framework, described succintly in Section 3, has already allowed us to obtain modular systems for all intuitionistic Scott-Lemmon logics [6]. In an attempt to make this system amenable for proof-search and decision procedures, we have started investigated a *fully nested* framework, presented in Section 4. We would be particularly interested in a suitable system for logic $\mathsf{IS4}$, whose decidability is not known; we discuss this direction in Section 5.

2 Intuitionistic modal logic

The language of intuitionisitic modal logic is the one of intuitionistic propositional logic with the modal operators $\Box$ and $\Diamond$. Starting with a set $\mathcal{A}$ of atomic propositions, denoted a, modal formulas are constructed from the grammar:

$$A ::= a \mid \bot \mid (A \wedge A) \mid (A \vee A) \mid (A \supset A) \mid \Box A \mid \Diamond A$$

The axiomatisation of intuitionistic modal logic IK [8,2] is obtained from intuitionistic propositional logic by adding:

- the *necessitation rule*: $\Box A$ is a theorem if A is a theorem; and
- the following five variants of the *distributivity axiom*:

$$\begin{array}{llllll} \mathsf{k}_1: & \Box(A \supset B) \supset (\Box A \supset \Box B) & \mathsf{k}_3: & \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) & \mathsf{k}_5: & \Diamond\bot \supset \bot \\ \mathsf{k}_2: & \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) & \mathsf{k}_4: & (\Diamond A \supset \Box B) \supset \Box(A \supset B) & & \end{array}$$

Definition 2.1 A *bi-relational frame* consists of a set of worlds W equipped with an accessibility relation R and a preorder $\leq$ satisfying:

(F_1) For $x, y, z \in W$, if xRy and $y \leq z$, there exists u s.t. $x \leq u$ and uRz.

(F_2) For $x, y, z \in W$, if $x \leq y$ and xRz, there exists u s.t. yRu and $z \leq u$.

Definition 2.2 A *bi-relational model* is a bi-relational frame with a monotone valuation function $V \colon W \to 2^{\mathcal{A}}$.

We write $x \Vdash a$ if $a \in V(x)$ and, by definition, it is never the case that $x \Vdash \bot$. The relation $\Vdash$ is extended to all formulas by induction, following the rules for both intuitionistic and modal Kripke models:

$$\begin{array}{lll} x \Vdash A \wedge B & \text{iff} & x \Vdash A \text{ and } x \Vdash B \\ x \Vdash A \vee B & \text{iff} & x \Vdash A \text{ or } x \Vdash B \\ x \Vdash A \supset B & \text{iff} & \text{for all } y \text{ with } x \leq y, \text{ if } y \Vdash A \text{ then } y \Vdash B \\ x \Vdash \Box A & \text{iff} & \text{for all } y \text{ and } z \text{ with } x \leq y \text{ and } yRz, z \Vdash A \\ x \Vdash \Diamond A & \text{iff} & \text{there exists a } y \text{ such that } xRy \text{ and } y \Vdash A \end{array} \tag{1}$$

Definition 2.3 A formula A is *valid* in a frame $\langle W, R, \leq \rangle$, if for all monotone valuations V and for all $w \in W$, we have $w \Vdash A$

Theorem 2.4 ([2,8]) *A formula A is a theorem of IK if and only if A is valid in every bi-relational frame.*

$$\mathsf{id}\ \frac{}{\mathcal{B}, x \leq y, \mathcal{L}, x{:}A \Rightarrow \mathcal{R}, y{:}A} \qquad \bot_{\mathsf{L}}\ \frac{}{\mathcal{B}, \mathcal{L}, x{:}\bot \Rightarrow \mathcal{R}}$$

$$\wedge_{\mathsf{L}}\ \frac{\mathcal{B}, \mathcal{L}, x{:}A, x{:}B \Rightarrow \mathcal{R}}{\mathcal{B}, \mathcal{L}, x{:}A \wedge B \Rightarrow \mathcal{R}} \qquad \wedge_{\mathsf{R}}\ \frac{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x{:}A \quad \mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x{:}B}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x{:}A \wedge B}$$

$$\vee_{\mathsf{L}}\ \frac{\mathcal{B}, \mathcal{L}, x{:}A \Rightarrow \mathcal{R} \quad \mathcal{B}, \mathcal{L}, x{:}B \Rightarrow \mathcal{R}}{\mathcal{B}, \mathcal{L}, x{:}A \vee B \Rightarrow \mathcal{R}} \qquad \vee_{\mathsf{R}}\ \frac{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x{:}A, x{:}B}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x{:}A \vee B}$$

$$\supset_{\mathsf{L}}\ \frac{\mathcal{B}, \mathcal{L}, x \leq y, y{:}A \Rightarrow \mathcal{R}, y{:}B}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x{:}A \supset B}\ y \text{ fresh}$$

$$\supset_{\mathsf{R}}\ \frac{\mathcal{B}, x \leq y, \mathcal{L} \Rightarrow \mathcal{R}, y{:}A \quad \mathcal{B}, x \leq y, \mathcal{L}, y{:}B \Rightarrow \mathcal{R}}{\mathcal{B}, x \leq y, \mathcal{L}, x{:}A \supset B \Rightarrow \mathcal{R}}$$

$$\Box_{\mathsf{L}}\ \frac{\mathcal{B}, x \leq y, yRz, \mathcal{L}, x{:}\Box A, z{:}A \Rightarrow \mathcal{R}}{\mathcal{B}, x \leq y, yRz, \mathcal{L}, x{:}\Box A \Rightarrow \mathcal{R}} \qquad \Box_{\mathsf{R}}\ \frac{\mathcal{B}, x \leq y, yRz, \mathcal{L} \Rightarrow \mathcal{R}, z{:}A}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x{:}\Box A}\ y, z \text{ fresh}$$

$$\Diamond_{\mathsf{L}}\ \frac{\mathcal{B}, xRy, \mathcal{L}, y{:}A \Rightarrow \mathcal{R}}{\mathcal{B}, \mathcal{L}, x{:}\Diamond A \Rightarrow \mathcal{R}}\ y \text{ fresh} \qquad \Diamond_{\mathsf{R}}\ \frac{\mathcal{B}, xRy, \mathcal{L} \Rightarrow \mathcal{R}, x{:}\Diamond A, y{:}A}{\mathcal{B}, xRy, \mathcal{L} \Rightarrow \mathcal{R}, x{:}\Diamond A}$$

$$\mathsf{refl}_{\leq}\ \frac{\mathcal{B}, x \leq x, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}} \qquad \mathsf{trans}_{\leq}\ \frac{\mathcal{B}, x \leq y, y \leq z, x \leq z, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, x \leq y, y \leq z, \mathcal{L} \Rightarrow \mathcal{R}}$$

$$\mathsf{F}_1\ \frac{\mathcal{B}, xRy, y \leq z, x \leq u, uRz, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, xRy, y \leq z, \mathcal{L} \Rightarrow \mathcal{R}}\ u \text{ fresh}$$

$$\mathsf{F}_2\ \frac{\mathcal{B}, xRy, x \leq z, y \leq u, zRu, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, xRy, x \leq z, \mathcal{L} \Rightarrow \mathcal{R}}\ u \text{ fresh}$$

Fig. 1. System $\mathsf{labIK}_{\leq}$

3 Fully labelled sequent calculus

Echoing the definition of bi-relational structures, we consider an extension of labelled deduction to the intuitionistic setting that uses two sorts of relational atoms, one for the modal accessibility relation R and another one for the intuitionistic preorder relation $\leq$ (similarly to [5] for epistemic logic).

Definition 3.1 A two-sided intuitionistic *fully labelled sequent* is of the form $\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}$ where $\mathcal{B}$ denotes a set of relational atoms xRy and preorder atoms $x \leq y$, and $\mathcal{L}$ and $\mathcal{R}$ are multi-sets of labelled formulas $x{:}A$ (for x and y taken from a countable set of labels and A an intuitionistic modal formula).

We obtain a proof system $\mathsf{labIK}_{\leq}$, displayed on Figure 1, for intuitionistic modal logic in this formalism. Most rules are similar to the ones of Simpson [10], but some are more explicitly in correspondence with the semantics by using the preorder atoms. For instance, the rules introducing the $\Box$-connective correspond to (1). Furthermore, our system must incorporate the conditions (F_1) and (F_2) into the deductive rules F_1 and F_2, and rules $\mathsf{refl}_{\leq}$ and $\mathsf{trans}_{\leq}$ are necessary to ensure that the preorder atoms behave as a preorder on labels.

$$\mathsf{id}\ \frac{}{\Gamma\{A^\bullet, A^\circ\}} \qquad \bot_{\mathsf{L}}\ \frac{}{\Gamma\{\bot^\bullet\}}$$

$$\wedge_{\mathsf{L}}\ \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \quad \wedge_{\mathsf{R}}\ \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}} \quad \vee_{\mathsf{L}}\ \frac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \vee B^\bullet\}} \quad \vee_{\mathsf{R}}\ \frac{\Gamma\{A^\circ, B^\circ\}}{\Gamma\{A \vee B^\circ\}}$$

$$\supset_{\mathsf{L}}\ \frac{\Gamma_1\{A \supset B^\bullet, A^\circ\} \quad \Gamma_1\{B^\bullet\}}{\Gamma_1\{A \supset B^\bullet\}} \quad \supset_{\mathsf{R}}\ \frac{\Gamma\{[\![A^\bullet, B^\circ]\!]\}}{\Gamma\{A \supset B^\circ\}}$$

$$\Box_{\mathsf{L}}\ \frac{\Gamma_1\{\Box A^\bullet, [A^\bullet, \Gamma_2]\}}{\Gamma_1\{\Box A^\bullet, [\Gamma_2]\}} \quad \Box_{\mathsf{R}}\ \frac{\Gamma\{[\![[A^\circ]]\!]\}}{\Gamma\{\Box A^\circ\}} \quad \Diamond_{\mathsf{L}}\ \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}} \quad \Diamond_{\mathsf{R}}\ \frac{\Gamma_1\{\Diamond A^\circ, [A^\circ, \Gamma_2]\}}{\Gamma_1\{\Diamond A^\circ, [\Gamma_2]\}}$$

$$\mathsf{mon}_{\mathsf{R}}\ \frac{\Gamma_1\{A^\circ, [\![A^\circ, \Gamma_2]\!]\}}{\Gamma_1\{[\![A^\circ, \Gamma_2]\!]\}} \quad \mathsf{mon}_{\mathsf{L}}\ \frac{\Gamma_1\{A^\bullet, [\![A^\bullet, \Gamma_2]\!]\}}{\Gamma_1\{A^\bullet, [\![\Gamma_2]\!]\}} \quad \mathsf{F}_1\ \frac{\Gamma_1\{[\Gamma_2], [\![[\Gamma_3]]\!]\}}{\Gamma_1\{[\Gamma_2, [\![\Gamma_3]\!]]\}}$$

Fig. 2. System $\mathsf{nIK}_\leq$

Theorem 3.2 *For any formula A, the following are equivalent.*

(i) *A is a theorem of* IK

(ii) *A is provable in* $\mathsf{labIK}_\leq + \mathsf{cut}$ *with* $\quad \mathsf{cut}\ \dfrac{\mathcal{B}_1, \mathcal{L} \Rightarrow \mathcal{R}, z{:}C \quad \mathcal{B}_2, \mathcal{L}, z{:}C \Rightarrow \mathcal{R}}{\mathcal{B}_1, \mathcal{B}_2, \mathcal{L} \Rightarrow \mathcal{R}}$

(iii) *A is provable in* $\mathsf{labIK}_\leq$

The proof is a careful adaptation of standard techniques (see [6] for details).

4 Fully nested sequent calculus

In standard nested sequent notation, brackets $[\cdot]$ are used to indicate the parent-child relation in the modal accessibility tree. $(\cdot)^\bullet$ and $(\cdot)^\circ$ annotations are used to indicate that the formulas would occur on the left-hand-side or right-hand-side of a sequent, respectively, in the absence of the sequent arrow.

To make it fully structured again, we enhance the structure with a second type of bracketting $[\![\cdot]\!]$ to encode the preorder relation.

Definition 4.1 A two-sided intuitionistic *fully nested sequent* is constructed from the grammar: $\Gamma ::= \varnothing \mid A^\bullet, \Gamma \mid A^\circ, \Gamma \mid [\Gamma] \mid [\![\Gamma]\!]$

The obtained nested sequent calculus $\mathsf{nIK}_\leq$ is displayed in Figure 2. The idea is similar to the fully labelled calculus but the shift of paradigm allowed us to make different design choices. In particular, the underlying tree-structure prevents us to express the rule F_2, but its absence is offset by the *monotonicity rules* mon_L and mon_R, which were admissible in $\mathsf{labIK}_\leq$. Another benefit of this addition is that rules $\mathsf{refl}_\leq$ and $\mathsf{trans}_\leq$ do not need any equivalent here.

5 Extensions: example of transitivity

As mentioned in the introduction, one of our motivation is to investigate decision procedure for axiomatic extensions of IK, for instance $\mathsf{IS4}$, intuitionistic logic of reflexive transitive frames. We will therefore illustrate our approach taking transitivity as a test-case.

The frame condition of transitivity ($\forall xyz.\, xRy \wedge yRz \supset xRz$) can be axiomatised by adding to IK the conjunction of the two versions of the $\mathsf{4}$-axiom:

$$\mathsf{4}_\Box : \Box A \supset \Box\Box A \qquad\qquad \mathsf{4}_\Diamond : \Diamond\Diamond A \supset \Diamond A$$

which are equivalent in classical modal logic. However, in intuitionistic modal logic they are not and they can be added to IK independently. From [8] we know they are in correspondence respectively with the following frame conditions:

$$\forall xyz.((xRy \wedge yRz) \supset \exists u.(x \leq u \wedge uRz)) \quad \forall xyz.((xRy \wedge yRz) \supset \exists u.(z \leq u \wedge xRu)) \quad (2)$$

Following Simpson [10] we could extend our basic sequent system for IK to $\mathsf{IK4} = \mathsf{IK} + (\mathsf{4}_\Box \wedge \mathsf{4}_\Diamond)$ with the rule

$$\mathsf{trans}_R \, \frac{\mathcal{B}, wRv, vRu, wRu, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, wRv, vRu, \mathcal{L} \Rightarrow \mathcal{R}}$$

Incorporating the preorder symbol into the syntax too, allowed us however to translate the conditions in (2) into separate inference rules for $\mathsf{4}_\Box$ and $\mathsf{4}_\Diamond$:

$$\mathsf{4}_\Box \, \frac{\mathcal{B}, xRy, yRz, uRz, x \leq u, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, xRy, yRz, \mathcal{L} \Rightarrow \mathcal{R}} \, u \text{ fresh} \qquad \mathsf{4}_\Diamond \, \frac{\mathcal{B}, xRy, yRz, xRu, z \leq u, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, xRy, yRz, \mathcal{L} \Rightarrow \mathcal{R}} \, u \text{ fresh}$$

These extensions for $\mathsf{labIK}_\leq$ are sound and complete; more generally, Theorem 3.2 can be extended to the class of intuitionisitc Scott-Lemmon logics [6].

Similar results for the fully nested sequent system are subject of ongoing study. Previous nested systems for intuitionistic modal logics [11,4] can be extended from IK to $\mathsf{IK4}$ by simply adding the following rules:

$$\Box_{\mathsf{L4}} \, \frac{\Gamma_1\{\Box A^\bullet, [\Box A^\bullet, \Gamma_2]\}}{\Gamma_1\{\Box A^\bullet, [\Gamma_2]\}} \qquad \Diamond_{\mathsf{R4}} \, \frac{\Gamma_1\{\Diamond A^\circ, [\Diamond A^\circ, \Gamma_2]\}}{\Gamma_1\{\Diamond A^\circ, [\Gamma_2]\}}$$

A great interest of such rules is their logical rather than structural nature, making them usually more suitable for proof-search procedures than their labelled counterpart.

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