A pure view of ecumenical modalities*

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Abstract. Recent works about ecumenical systems, where connectives from classical and intuitionistic logics can co-exist in peace, warmed the discussion of proof systems for combining logics. This discussion has been extended to alethic K-modalities: using Simpson's meta-logical characterization, necessity is independent of the viewer, while possibility can be either *intuitionistic* or *classical*. In this work, we propose an *internal pure* calculus for ecumenical modalities, nEK, where every basic object of the calculus can be read as a formula in the language of the ecumenical modal logic EK. We prove that nEK is sound and complete w.r.t. the ecumenical birrelational semantics, and study fragments and modal extensions.

1 Introduction

Ecumenism can be seen as the search for *unicity*, where different thoughts, ideas or points of view can harmonically co-exist. In mathematical logic, ecumenical approaches for a peaceful coexistence of different logical systems have been studied deeply, *e.g.* [Gir93,LM11].

More recently, Prawitz proposed a natural deduction system sharing classical and intuitionistic connectives [Pra15]. The fundamental question he addressed was: what makes a connective classical or intuitionistic? We will illustrate, with a simple example, some ways of answering this. Consider the following statement, where $x, y, z \in \mathbb{R}$ and $z \geq 0$:

$$\text{if } x + y = 2z \text{ then } x \geq z \text{ or } y \geq z.$$

How should we interpret "if then" and "or" in this sentence, so that it will be valid? The answer is: it depends! We could certainly interpret both classically, and this would satisfy a classical mathematician (CM), but it would rule out intuitionistic mathematicians (IM). Since intuitionists can see classical tautologies through the lens of double negation, we could embed this classical interpretation in the intuitionistic setting as:

$$\text{not (not (if } x + y = 2z \text{ then } x \geq z \text{ or } y \geq z)).$$

This would indeed make CM and IM happy.

But a finer analysis shows that, while the disjunction should definitely be classical for guaranteeing the validity of the sentence, the implication can have a perfect intuitionistic interpretation. That is, the statement can be understood by CM and IM as:

$$\text{if } x + y = 2z \text{ then not (not (} x \geq z \text{ or } y \geq z)).$$

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Prawitz' ecumenism can be summarized as: pinpoint the exact places where classical and intuitionistic world's views differ and add there ecumenical glasses. The example above shows that CM and IM consider, in fact, different connectives for disjunction: $\vee_c$ and $\vee_i$, respectively. Now, what about the other first-order connectives? Prawitz answered this question by presenting an ecumenical natural deduction system.

In [PPdP19], we justified some of Prawitz' choices via pure proof theoretical reasoning, using sequent based systems. Consider the well known classical and intuitionistic sequent systems G3c and G3i [TS96]. Since all rules in G3c are invertible, no choices have to be made during a pure classical proof: one can apply any rule in any order. This is not the case in G3i: choices may have to be made for disjunction, implication, and existential quantifier. This suggests that CM and IM would share the universal quantifier, conjunction, and the constant for the absurd (hence also negation) – *the neutral connectives*, but they would each have their own existential quantifier, disjunction, and implication, with different meanings.

Under this discussion, our simple statement is ecumenically translated as

$$(x + y = 2z) \rightarrow_i x \geq z \vee_c y \geq z.$$

Now the classical mathematician would see everything just fine (since she cannot differ classical from intuitionistic), while the intuitionistic mathematician would put her ecumenical glasses only when it comes to see the disjunction. And they would both be happy and agree on the statement. *This* is the essence of ecumenism!

In [MPPS20], we have extended this discussion to modalities. Since alethic modalities are interpreted as "necessity" and "possibility", with this extension of the notion of truth, how would CM and IM view such concepts? Using Simpson's meta-logical characterization [Sim94], the answer is that, if something is necessarily true, then it is independent of the viewer. Possibility, on the other hand, can be either *intuitionistic:* in the sense that one should have a guarantee that something will eventually be true; or *classical:* in the sense that it is not the case that necessarily something will not be true. Hence CM and IM share the necessity connective $\Box$, but each would have its own possibility views, represented by $\Diamond_c$ and $\Diamond_i$, respectively.

But our solution was not entirely satisfactory since the ecumenical calculi presented so far are not *pure*: the introduction rules for some connectives heavily depend on negation and other connectives. Moreover, the ecumenical modal systems in [MPPS20] are *external*: the basic objects are formulas of a more expressive language which imports or partially encodes the logic's semantics.

This paper is devoted to tackle these problems, proposing an *internal pure* calculus for ecumenical modalities, where every basic object of the calculus can be read as a formula in the language of the logic. For that, we will use *nested systems* [Brü09,Pog09] with a *stoup* [Gir91], together with a notion of *polarized ecumenical formulas*. Nested systems are extensions of the sequent framework where each sequent is replaced by a tree of sequents. The stoup is a distinguished context containing a single formula. Finally, polarized formulas can be *negative* if the main connective is classical or the negation, or *positive* otherwise. The idea is that negative formulas are stored in the classical context, while positive formulas are decomposed in the stoup. This not only allows for establishing the meaning of modalities via the rules that determine their correct use (logical inferentialism), but it also places ecumenical systems as unifying frameworks for modalities, where well known modal systems appear as fragments.

Organization and contributions. Sec. 2 presents the notation for modal formulas, the labeled system labEK and the ecumenical birelational semantics; Sec 3 introduces the ecumenical nested system nEK and its normalization procedure; in Secs. 4 and 5 soundness and completeness of nEK w.r.t. the ecumenical birelational semantics are proved; Sec. 6 identifies the classical and intuitionistic fragments of nEK; Sec 7 presents modal extensions; and Sec. 8 concludes the paper. Appendices A, B and C show all the proof systems addressed in this paper, the key cases of the proof of cut-elimination, and an alternative proof of soundness based on an internal interpretation of nestings.

2 Preliminaries

In [MPPS20] we have proposed an ecumenical version of normal modal logic, where classical and intuitionistic modalities co-exist in the same system. The system adopts Simpson's approach [Sim94], where a modal logic is characterized by the respective interpretation of the modal model in the (first-order) meta-theory, called meta-logical characterization. Hence modalities are translated into the (ecumenical) first-order logic LE [Pra15,PPdP19], justified by the interpretation of alethic modalities in a Kripke model. The interesting aspect of this ecumenical interpretation is that the presence of classical and intuitionistic existential connectives in LE induces two *possibility* modalities, while the neutral universal quantifier in LE entails a neutral *necessity* modality.

The language $\mathcal{A}$ used for ecumenical modal systems is described as follows. Formulas are generated by the following grammar:

$$A ::= p_i \mid p_c \mid \bot \mid \neg A \mid A \wedge A \mid A \vee_i A \mid A \vee_c A \mid A \rightarrow_i A \mid A \rightarrow_c A \mid \Box A \mid \Diamond_i A \mid \Diamond_c A$$

We will use a subscript c for the classical meaning and i for the intuitionistic one, dropping such subscripts when formulas/connectives can have either meaning. Classical and intuitionistic propositional variables ($p_c, p_i, \ldots$) co-exist in $\mathcal{A}$ but have different meanings. The neutral logical connectives $\{\bot, \neg, \wedge, \Box\}$ are common for classical and intuitionistic fragments, while $\{\rightarrow_i, \vee_i, \Diamond_i\}$ and $\{\rightarrow_c, \vee_c, \Diamond_c\}$ are restricted to intuitionistic and classical interpretations, respectively. A formula is called *negative* if its main connective is classical or the negation, and *positive* otherwise.

The meta-logical characterization naturally induces a labeled proof system [Sim94]. The language $\mathcal{L}$ of *labeled modal formulas* is determined by *labeled formulas* of the form $x : A$ with $A \in \mathcal{A}$ or *relational atoms* of the form xRy, where x, y range over a set of variables. *Labeled sequents* have the form $\Gamma \Rightarrow x : A$, where Γ is a multiset containing labeled modal formulas and relational atoms. In what follows, if L is a sequent based calculus, we use $\vdash_{\mathsf{L}} \Gamma \Rightarrow A$ to denote that there is an L-proof of $\Gamma \Rightarrow A$. The labeled ecumenical system labEK [MPPS20] is presented in Figure 1 in Appendix A.

Example 1. Below the derivation in labEK of the distributivity of the diamond w.r.t the disjunction (see axiom k_2 in Section 5).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{xRy, y : A \Rightarrow y : A}\,\mathsf{init}}{xRy, y : A \Rightarrow x : \Diamond_i A}\,\Diamond_i R}{xRy, y : A \Rightarrow x : \Diamond_i A \vee_i \Diamond_i B}\,\vee_i R \qquad \dfrac{\dfrac{\dfrac{}{xRy, y : B \Rightarrow y : B}\,\mathsf{init}}{xRy, y : B \Rightarrow x : \Diamond_i B}\,\Diamond_i R}{xRy, y : B \Rightarrow x : \Diamond_i A \vee_i \Diamond_i B}\,\vee_i R}{\dfrac{xRy, y : A \vee_i B \Rightarrow x : \Diamond_i A \vee_i \Diamond_i B}{x : \Diamond_i (A \vee_i B) \Rightarrow x : \Diamond_i A \vee_i \Diamond_i B}\,\Diamond_i L}\,\vee_i L}{\Rightarrow x : \Diamond_i (A \vee_i B) \rightarrow_i (\Diamond_i A \vee_i \Diamond_i B)}\,\rightarrow_i R$$

2.1 Ecumenical birelational models

The ecumenical birelational Kripke semantics, which is an extension of the proposal in [PR17] to modalities, was presented in [MPPS20].

Definition 1. *A* birelational Kripke model *is a quadruple* $\mathcal{M} = (W, \leq, R, V)$ *where* (W, R, V) *is a Kripke model such that* W *is partially ordered with order* $\leq$*,* $R \subset W \times W$ *is a binary relation, the satisfaction function* $V : \langle W, \leq\rangle \to \langle 2^{\mathcal{P}}, \subseteq\rangle$ *is monotone and:*
F1. For all worlds w, v, v'*, if* wRv *and* $v \leq v'$*, there is a* w' *such that* $w \leq w'$ *and* $w'Rv'$*;*
F2. For all worlds w', w, v*, if* $w \leq w'$ *and* wRv*, there is a* v' *such that* $w'Rv'$ *and* $v \leq v'$*.*

An ecumenical modal Kripke model *is a birelational Kripke model such that truth of an ecumenical formula at a point* w *is the smallest relation* $\models_{\mathsf{E}}$ *satisfying*

$\mathcal{M}, w \models_{\mathsf{E}} p_i$	*iff*	$p_i \in V(w)$;
$\mathcal{M}, w \models_{\mathsf{E}} A \wedge B$	*iff*	$\mathcal{M}, w \models_{\mathsf{E}} A$ *and* $\mathcal{M}, w \models_{\mathsf{E}} B$;
$\mathcal{M}, w \models_{\mathsf{E}} A \vee_i B$	*iff*	$\mathcal{M}, w \models_{\mathsf{E}} A$ *or* $\mathcal{M}, w \models_{\mathsf{E}} B$;
$\mathcal{M}, w \models_{\mathsf{E}} A \to_i B$	*iff*	*for all* v *such that* $w \leq v$, $\mathcal{M}, v \models_{\mathsf{E}} A$ *implies* $\mathcal{M}, v \models_{\mathsf{E}} B$;
$\mathcal{M}, w \models_{\mathsf{E}} \neg A$	*iff*	*for all* v *such that* $w \leq v$, $\mathcal{M}, v \not\models_{\mathsf{E}} A$;
$\mathcal{M}, w \models_{\mathsf{E}} \bot$		*never holds*;
$\mathcal{M}, w \models_{\mathsf{E}} \Box A$	*iff*	*for all* v, w' *such that* $w \leq w'$ *and* $w'Rv$, $\mathcal{M}, v \models_{\mathsf{E}} A$.
$\mathcal{M}, w \models_{\mathsf{E}} \Diamond_i A$	*iff*	*there exists* v *such that* wRv *and* $\mathcal{M}, v \models_{\mathsf{E}} A$.
$\mathcal{M}, w \models_{\mathsf{E}} p_c$	*iff*	$\mathcal{M}, w \models_{\mathsf{E}} \neg(\neg p_i)$;
$\mathcal{M}, w \models_{\mathsf{E}} A \vee_c B$	*iff*	$\mathcal{M}, w \models_{\mathsf{E}} \neg(\neg A \wedge \neg B)$;
$\mathcal{M}, w \models_{\mathsf{E}} A \to_c B$	*iff*	$\mathcal{M}, w \models_{\mathsf{E}} \neg(A \wedge \neg B)$.
$\mathcal{M}, w \models_{\mathsf{E}} \Diamond_c A$	*iff*	$\mathcal{M}, w \models_{\mathsf{E}} \neg\Box\neg A$.

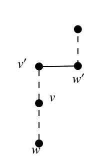
We say that a formula A *is* valid *in a model* $\mathcal{M} = (W, \leq, R, V)$ *if for all* $w \in W$ *we have* $w \models_{\mathsf{E}} A$*. A formula* A *is* valid *in a frame* $\langle W, \leq, R\rangle$ *if, for all valuations* V*,* A *is valid in the model* $(W, \leq, R, V)$*. Finally, we say a formula is* valid*, if it is valid in all frames.*

Since, restricted to intuitionistic and neutral connectives, $\models_{\mathsf{E}}$ is the usual birelational interpretation $\models$ for IK [Sim94], and since the classical connectives are interpreted via the neutral ones using the double-negation translation, an ecumenical modal Kripke model coincides with the standard birelational Kripke model for intuitionistic modal logic IK. Hence the following result easily holds from the similar result for IK.

Theorem 1 ([MPPS20]). *The system* labEK *is sound and complete w.r.t. the ecumenical modal Kripke semantics, that is,* $\vdash_{\mathsf{labEK}} x : A$ *iff* $\models_{\mathsf{E}} A$*.*

Remark 1. It is interesting to note that the Kripke semantics for the classical connectives is more complex than the respective intuitionistic ones. In fact, the definition of $\models_{\mathsf{E}}$ for the classical diamond is equivalent to

$$\mathcal{M}, w \models_{\mathsf{E}} \Diamond_c A \text{ iff } \forall v \geq w. \exists u. v\, (\leq \circ R \circ \leq)\, u,\ \mathcal{M}, u \models_{\mathsf{E}} A$$



where $v\,(\leq \circ R \circ \leq)\,u$ represents that there exist $v', w' \in W$ such that $v \leq v'$, $v'Rw'$ and $w' \leq u$. Although intriguing, this kind of two-level semantics also appears in the Kripke model for classical logic in [ILH10], where the forcing relation is defined on the top of the primitive notion of "strong refutation".

3 A nested system for ecumenical modal logic

The two main criticisms that can be done regarding the system labEK are: (i) it is *external*, in the sense that it includes some semantics in the technical machinery, hence deriving statements that are not always purely logical formulas; and (ii) it is not *pure*, in the sense that negation still plays an important role on interpreting classical connectives. For example, the introduction of the classical diamond (rule $\lozenge_c R$ in Figure 1) depends on its boxed negated version.

This section is devoted to tackle all such points and propose an *internal pure* calculus for ecumenical modalities, where every basic object of the calculus can be read as a formula in the language of the logic, with no use of auxiliary negations.

The inspiration comes from Girard's notion of *stoup* [Gir91] and Straßburger's *nested system* for IK [Str13]. The main idea is to pass sequent systems of the form $\Sigma \Rightarrow \Pi$, with Σ, Π multisets of formulas, through a two-phase refinement: the first one is to separate the succedent Π into two parts: one that is essentially classical; and another containing a single formula, the stoup. The second one is to add nested layers to sequents, which intuitively corresponds to worlds in the Kripke structure [Fit14].

The structure of a nested sequent for ecumenical modal logics is hence a tree whose nodes are multisets of formulas, just like in [Str13,CMS16]. The difference is that the ecumenical formulas can be *left inputs* (in the left contexts – marked with a full circle $^\bullet$), *right inputs* (in the classical right contexts – marked with a triangle $^\triangledown$) or a *single right output* (the stoup – marked with a white circle $^\circ$).

Definition 2. Ecumenical nested sequents *are defined in terms of a grammar of* input sequents *(written Λ) and* full sequents *(written Γ) where the left/right input formulas are denoted by $A^\bullet$ and $A^\triangledown$, respectively, and A° denote output formulas. When the distinction between input and full sequents is not essential or cannot be made explicit, we will use Δ to stand for either case. The relationship between parent and child in the tree will be represented using bracketing* $[\cdot]$.

$$\Lambda \coloneqq \varnothing \mid A^\bullet, \Lambda \mid A^\triangledown, \Lambda \mid [\Lambda] \qquad \Gamma \coloneqq A^\circ, \Lambda \mid [\Gamma], \Lambda \qquad \Delta \coloneqq \Lambda \mid \Gamma$$

We write $\Gamma^{\perp^\circ}$ for the result of replacing an output formula from Γ by $\perp^\circ$, while $\Lambda^{\perp^\circ}$ represents the result of adding anywhere of the input context Λ the output formula $\perp^\circ$. Finally, Δ^* is the result of erasing an output formula (if any) from Δ.

Example 2. The nested sequent $\lozenge_c A^\triangledown, [\neg A^\circ]$ represents a tree of sequents where $\lozenge_c A$ is in the right (classical) input context of the root sequent, while $\neg A$ is in the output context (stoup), in the leaf sequent.

Observe that full sequents Γ necessarily have exactly one output-like formula, having the form

$$\Lambda_1, [\Lambda_2, [\ldots, [\Lambda_n, A^\circ]] \ldots]$$

As usual, we allow sequents to be empty, and we consider sequents to be equal modulo associativity and commutativity of the comma.

The next definition (of contexts) allows for identifying subtrees, necessary for introducing inference rules for nested sequents.

Definition 3. *An n-*ary context *$\Delta\left\{^1\right\}\cdots\left\{^n\right\}$ is like a sequent but contains n pairwise distinct numbered* holes { } *wherever a formula may otherwise occur.*

Given n sequents $\Delta_1, \ldots, \Delta_n$, we write $\Delta\{\Delta_1\}\cdots\{\Delta_n\}$ for the sequent where the i-th hole in $\Delta\left\{^1\right\}\cdots\left\{^n\right\}$ has been replaced by Δ_i (for $1 \leq i \leq n$), assuming that the result is well-formed, i.e.*, there is at most one output formula. If $\Delta_i = \emptyset$ the hole is removed.*

A full context *is a context of the form $\Gamma\left\{^1\right\}\cdots\left\{^n\right\}$, while an* input context *is of the form $\Lambda\left\{^1\right\}\cdots\left\{^n\right\}$.*

Given two nested sequents with a hole $\Gamma^i\{\} = \Delta^i_1, \left[\Delta^i_2, \left[\ldots, \left[\Delta^i_n, \{\}\right]\right]\ldots\right]$, $i \in \{1,2\}$, their merge *is the nested sequent with a hole*[5]

$$\Gamma^1 \otimes \Gamma^2\{\} = \Delta^1_1, \Delta^2_1, \left[\Delta^1_2, \Delta^1_2, \left[\ldots, \left[\Delta^1_n, \Delta^2_n, \{\}\right]\right]\ldots\right]$$

Figure 2 in Appendix A presents the system nEK, a nested sequent system for the ecumenical modal logic EK.

Example 3. Below left is the nested derivation corresponding to the labeled derivation in Example 1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[A^\bullet, A^\circ]}\,\text{init}}{\Diamond_i A^\circ, [A^\bullet]}\,\Diamond_i^\circ}{\Diamond_i A \vee_i \Diamond_i B^\circ, [A^\bullet]}\,\vee_i^\circ \quad \dfrac{\dfrac{\dfrac{}{[B^\bullet, B^\circ]}\,\text{init}}{\Diamond_i B^\circ, [B^\bullet]}\,\Diamond_i^\circ}{\Diamond_i A \vee_i \Diamond_i B^\circ, [B^\bullet]}\,\vee_i^\circ}{\dfrac{\Diamond_i A \vee_i \Diamond_i B^\circ, [A \vee_i B^\bullet]}{\Diamond_i (A \vee_i B)^\bullet, \Diamond_i A \vee_i \Diamond_i B^\circ}\,\Diamond_i^\bullet}\,\vee_i^\bullet}{\Diamond_i (A \vee_i B) \rightarrow_i (\Diamond_i A \vee_i \Diamond_i B)^\circ}\,\rightarrow_i^\circ \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[A^\bullet, A^\triangledown, \bot^\circ]}\,\text{init}_\text{c}}{[A^\triangledown, \neg A^\circ]}\,\neg^\circ}{\Diamond_c A^\triangledown, [\neg A^\circ]}\,\Diamond_c^\triangledown}{\Box\neg A^\circ, \Diamond_c A^\triangledown}\,\Box^\circ}{\neg\Box\neg A^\bullet, \Diamond_c A^\triangledown, \bot^\circ}\,\neg^\bullet}{\neg\Box\neg A^\bullet, \Diamond_c A^\circ}\,\text{store}$$

The derivation above right shows part of the proof that $\Diamond_c$ can be defined from $\Box$ ($\Diamond_c A \equiv \neg\Box\neg A$). Note the instance of the classical general version of the initial axiom, init_c (see Theorem 2 in the next section). It also *illustrates* well the relationship between nestings, classical inputs, and Kripke structures: reading the proof bottom-up, the store rule is a delay on applying rules over classical connectives. It corresponds to moving the formula up w.r.t. $\leq$ in the Kripke birrelational semantics. The rule $\Box^\circ$, on the other hand, slides the formula to a fresh new world, related to the former one through the relation R. Finally, rule $\neg^\circ$ also moves up the formula w.r.t. $\leq$. Compare this description with the image in Remark 1. In this paper, we will not explore *formally* the relationship between delays/negations/nestings and the Kripke semantics.

We prove next that nEK is *harmonic*, that is, it has the identity expansion and cut-elimination properties.

3.1 Harmony

A logical connective is called *harmonious* in a certain proof system if there exists a certain balance between the rules defining it. For example, in natural deduction based

[5] As observed in [Lel19], the merge is a “zipping” of the two nested sequents along the path from the root to the hole.

systems, harmony is ensured when introduction/elimination rules do not contain insufficient/excessive amounts of information [DD20]. In sequent calculus, this property is often guaranteed by the admissibility of a general initial axiom (*identity-expansion*) and the cut rule (*cut-elimination*) [MP13]. In the following, we will prove harmony, together with some intermediate results. We start with a proof theoretical result in nEK, which has a standard proof (see [PPdP19] and [MPPS20] for similar results).

Lemma 1. *In* nEK*, the rules* $\vee_c^\bullet, \vee_c^\triangledown, \rightarrow_c^\bullet, \rightarrow_c^\triangledown, \neg^\bullet, \neg^\circ, p_c^\bullet, p_c^\triangledown, \lozenge_c^\bullet, \lozenge_c^\triangledown$ *and* D *are* invertible*, that is, in any application of such rules, if the conclusion is a provable nested sequent so are the premises. The rules* $\wedge^\bullet, \wedge^\circ, \vee_i^\bullet, \rightarrow_i^\circ, \lozenge_i^\bullet, \square^\bullet, \square^\circ$ *and* store *are* totally invertible*, that is, they are invertible and do not have restrictions over contexts.*

Observe that the invertible but not totally invertible rules in nEK concern negative formulas, hence they can only be applied in the presence of empty ($\bot^\circ$) stoups. Note also that the rules W, $\vee_i^\circ$, and $\lozenge_i^\circ$ are not invertible, while $\rightarrow_i^\bullet$ is invertible only w.r.t. the right premise.

Theorem 2. *The following rules are admissible in* nEK

$$\frac{}{\Lambda\{A^\bullet, A^\circ\}}\ \mathsf{init_g} \qquad \frac{}{\Gamma^{\bot^\circ}\{A^\bullet, A^\triangledown\}}\ \mathsf{init_c} \qquad \frac{\Gamma}{\Lambda \otimes \Gamma}\ \mathsf{W}_c \qquad \frac{\Lambda \otimes \Lambda \otimes \Gamma}{\Lambda \otimes \Gamma}\ \mathsf{C}_c$$

Proof. The proofs are by standard induction on the height of derivations. The proof of admissibility of W_c does not depend on any other result, while the admissibility of C_c depends on invertibility results (Lemma 1). The proof of admissibility of the general initial axioms is by mutual induction. Bellow we show the modal cases where, by inductive hypothesis, instances of the axioms hold for the premises.

$$\cfrac{\cfrac{\cfrac{}{\Gamma^{\bot^\circ}\{[A^\bullet, A^\triangledown]\}}\ \mathsf{init_c}}{\Gamma^{\bot^\circ}\{\lozenge_c A^\triangledown, [A^\bullet]\}}\ \lozenge_c^\triangledown}{\Gamma^{\bot^\circ}\{\lozenge_c A^\bullet, \lozenge_c A^\triangledown\}}\ \lozenge_c^\bullet \qquad \cfrac{\cfrac{\cfrac{}{\Lambda\{[A^\bullet, A^\circ]\}}\ \mathsf{init_g}}{\Lambda\{\square A^\bullet, [A^\circ]\}}\ \square^\bullet}{\Lambda\{\square A^\bullet, \square A^\circ\}}\ \square^\circ \quad \square$$

Proving admissibility of cut rules in sequent based systems with multiple contexts is often tricky, since the cut formulas can change contexts during cut reductions. This is the case for nEK. The output cut rules shown bellow are admissible in nEK.

Theorem 3. *The following intuitionistic and classical cut rules are admissible in* nEK

$$\frac{\Lambda\{P^\circ\} \quad \Gamma\{P^\bullet\}}{\Lambda \otimes \Gamma\{\emptyset\}}\ \mathsf{cut}^\circ \qquad \frac{\Lambda^{\bot^\circ}\{N^\triangledown\} \quad \Gamma\{N^\bullet\}}{\Lambda \otimes \Gamma\{\emptyset\}}\ \mathsf{cut}^\triangledown$$

Proof. The dynamic of the proof is the following: cut applications either move up in the proof, or are substituted by simpler cuts of the same kind, as in usual cut-elimination reductions. The cut applications swing from intuitionistic to classical and vice-versa in the principal cases, when the polarity of the subformulas flip. All such moves either decrease the height of the application of the cut rule (called cut-height) or the complexity of the formula under cut (called ecumenical weight of the cut-formula). When the cut instance touches a leave, it is eliminated. The definition of ecumenical weight and the proofs for some critical cases are shown in Appendix B. $\square$

4 Soundness

In this section we will show that all rules presented in Figure 2 are sound w.r.t. the ecumenical birelational Kripke model.

The idea is to prove that the rules of the system nEK preserves *validity*, in the sense that if the interpretation of the premises is valid, so is the interpretation of the conclusion. The first step is to determine the interpretation of ecumenical nested sequents. As shown in the Appendix C, it is possible to interpret nested sequents as ecumenical modal formulas, and hence prove soundness in the same way as in [Str13]. This is really interesting, since it shows a direct interpretation of nested sequents as ecumenical formulas, which reveals that nEK is an *internal proof system*.

In this section, we will rather present the translation of nestings to labeled sequents, hence establishing, at the same time, soundness of nEK and the relation between this system with labEK.

First of all, we observe that the entailment in ecumenical systems is intrinsically intuitionistic, in the sense that $\Gamma \Rightarrow B$ in LE iff $\vdash_{\mathsf{LEci}} \bigwedge \Gamma \rightarrow_i B$ (see [PPdP19]). Moreover, the classical connectives are defined via the intuitionistic ones by sporadic double-negation. Another interesting aspect is that, in the labelled ecumenical modal system labEK, fresh world labels can be created (bottom-up) by the box operator in succedents and both diamond connectives in antecedents. Yet, once this new world is created, it is shared by all modal formulas, independently of the intuitionistic or classical nature.

This suggests the following interpretation of nested into labeled ecumenical sequents.

Definition 4. *Let $\Pi^{\bullet}, \Pi^{\triangledown}, \Pi^{\circ}$ represent that all formulas in the respective contexts are input left, right, or output formulas, respectively. The underlying multiset Π will represent the formulas in $\mathcal{A}$ corresponding to the respective unmarked formula. The translation $[\![\cdot]\!]_x$ from nested into labeled sequents is defined recursively by*

$$[\![\Pi_1^{\bullet}, \Pi_2^{\triangledown}, \Pi_3^{\circ}, [\Delta_1], \ldots, [\Delta_n]]\!]_x := (\{xRx_i\}_i, x : \Pi_1, x : \neg\Pi_2 \Rightarrow x : \Pi_3) \otimes_x \{[\![\Delta_i]\!]_{x_i}\}_i$$

where $1 \leq i \leq n$, x_i are fresh, and the merge operation on labeled sequents is defined as

$$(\Sigma_1 \Rightarrow \Pi_1) \otimes_x (\Sigma_2 \Rightarrow \Pi_2) \ := \ \Sigma_1, \Sigma_2 \Rightarrow \Pi_1, \Pi_2$$

Given $\mathcal{R}$ a set of relational formulas, we will denote by xR^*z the fact that there is a path from x to z in $\mathcal{R}$, i.e., there are $y_j \in \mathcal{R}$ for $0 \leq j \leq k$ such that $x = y_0, y_{j-1}Ry_j$ and $y_k = z$.

That is, right input formulas are translated as negated left input formulas, and nestings correspond to worlds in the Kripke structure. The next result shows that, in fact, this interpretation is correct.

Theorem 4. *If $\vdash_{\mathsf{nEK}} \Gamma$ then $\vdash_{\mathsf{labEK}} [\![\Gamma]\!]_x$.*

Proof. The proof is by standard induction on the proof π of Γ. We will illustrate a classical and a modal case.

– If the last rule applied in π is $(\vee_c^{\triangledown})$, by inductive hypothesis,

$$[\![\Gamma^{\perp^{\circ}}\{A^{\triangledown}, B^{\triangledown}\}]\!]_x = \mathcal{R}, \Sigma, z : \neg A, z : \neg B \Rightarrow x : \bot$$

is provable for a set $\mathcal{R}$ and a multiset Σ of relational and labeled formulas, respectively, translating the context $\Gamma^{\perp^\circ}$, where xR^*z. Hence:

$$\dfrac{\dfrac{\mathcal{R}, \Sigma, z : \neg A, z : \neg B \Rightarrow z : \bot}{\mathcal{R}, \Sigma \Rightarrow z : A \vee_c B}\ \vee_c R}{\mathcal{R}, \Sigma, z : \neg(A \vee_c B) \Rightarrow x : \bot}\ \neg L$$

Observe that, due to the rule W in labEK, the label of right $\bot$ is irrelevant.

– If the last rule applied in π is $(\Diamond_c^{\nabla})$, by inductive hypothesis,

$$[\![\Delta_1^{\perp^\circ}\left\{\Diamond_c A^{\nabla}, \left[A^{\nabla}, \Delta_2^{\perp^\circ}\right]\right\}]\!]_x = \mathcal{R}, zRy, \Sigma, z : \neg(\Diamond_c A), y : \neg A \Rightarrow x : \bot$$

is provable for a set $\mathcal{R}$ and a multiset Σ of relational and labelled formulas, respectively, translating the contexts $\Delta_1^{\perp^\circ}, \Delta_2^{\perp^\circ}$, where xR^*z. Hence:

$$\dfrac{\dfrac{\dfrac{\mathcal{R}, zRy, \Sigma, z : \neg(\Diamond_c A), z : \Box\neg A, y : \neg A \Rightarrow z : \bot}{\mathcal{R}, zRy, \Sigma, z : \neg(\Diamond_c A), z : \Box\neg A \Rightarrow z : \bot}\ \Box L}{\mathcal{R}, zRy, \Sigma, z : \neg(\Diamond_c A) \Rightarrow z : \Diamond_c A}\ \Diamond_c R}{\mathcal{R}, zRy, \Sigma, z : \neg(\Diamond_c A) \Rightarrow x : \bot}\ \neg L$$

□

The proof above also establishes the relationship between *proofs* in nEK and labEK: the right input context stores *negative* formulas, which are in fact negated positive formulas (as in Girard's LC [Gir91]), and the decision rule D in nEK is mimicked in labEK by applications of the left rule for negation. In this way, the use of nestings together with decision and store rules imposes a *discipline* on rule applications in labeled systems.

Theorems 1 and 4 immediately imply the following.

Corollary 1. *The nested system* nEK *is sound w.r.t. ecumenical modal Kripke semantics.*

Finally, we observe that translating sequents and proofs from labeled to nested systems is not a simple task, sometimes even impossible. Hence the method described above often does not work for proving completeness of the system. We will show completeness with respect to the *Hilbert system*.

5 Completeness

Classical modal logic K is characterized as propositional classical logic, extended with the *necessitation rule* (presented in Hilbert style) $A/\Box A$ and the *distributivity axiom* $\mathsf{k} :\ \Box(A \to B) \to (\Box A \to \Box B)$.

In the intuitionistic case, however, the discussion is more interesting, since there are many variants of axiom k that induces classically, but not intuitionistically, equivalent systems (see [PS86,Sim94]). In fact, the following axioms classically follow from k and the De Morgan laws, but not in an intuitionistic setting

$$\mathsf{k}_1 :\ \Box(A \to B) \to (\Diamond A \to \Diamond B) \qquad \mathsf{k}_2 :\ \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$$
$$\mathsf{k}_3 :\ (\Diamond A \to \Box B) \to \Box(A \to B) \qquad \mathsf{k}_4 :\ \Diamond\bot \to \bot$$

Combining axiom k with axioms $k_1 - k_4$ characterizes intuitionistic modal logic IK [PS86].

In the ecumenical setting, this discussion is even more interesting, since there are many more variants of k, depending on the classical or intuitionistic interpretation of implications and diamonds.

The theorems in *ecumenical modal logic* EK are defined as being the formulas that are derivable from the axioms of intuitionistic propositional logic plus the definitions of classical operators using negation, and the intuitionistic versions of the axioms $k - k_4$. It is easy to see that a formula is derivable in EK iff it is valid in all birelational Kripke frames (see [MPPS20]).

Also, it is a trivial matter to check that the EK axioms are provable in nEK (see *e.g.* Example 3). Hence, in the presence of cut-elimination (Section 3.1), the completeness result is valid for nEK.

Theorem 5. *Every theorem of the logic* EK *is provable in* nEK*.*

6 Extracting fragments

In this section, we will study pure classical and intuitionistic fragments of nEK. For the sake of simplicity, negation will not be considered a primitive connective, it will rather take its respective intuitionistic or classical form.

Definition 5. *An ecumenical modal formula C is* classical *(*intuitionistic*) if it is built from classical (intuitionistic) atomic propositions using only neutral and classical (intuitionistic) connectives but negation, which will be replaced by $A \rightarrow_c \bot$ ($A \rightarrow_i \bot$).*

The first thing to observe is that, when only pure fragments are concerned, weakening is admissible. Observe that this is not the case for the whole system nEK. In fact, $A \vee_c \neg A^{\triangledown}, C^{\circ}$ is provable in nEK for any formula C, but the proof necessarily starts with an application of the rule W if, *e.g.*, C is an atomic formula p_i.

Proposition 1. *Let nEK_i (nEK_c) be the system obtained from* nEK – W *by restricting the rules to the intuitionistic (classical) case (see Figures 3 and 4 in Appendix A). The rule* W *is admissible in nEK_i and nEK_c.*

Proof. For the intuitionistic fragment, the proof is standard, by induction on the height of derivations (considering all possible rule applications). The classical case is more involved. The idea is that classical formulas in the *stoup* are eagerly decomposed until either an axiom is applied, or the formula is stored in the classical input context and the stoup becomes empty. This is only possible because the rules $\wedge^{\circ}$ and $\square^{\circ}$ are totally invertible and all the other rules in nEK_c are invertible (Lemma 1). Formally, the following discipline is complete for nEK_c, when proving a nested sequent Γ:

i. Apply the rules $\wedge^{\bullet}, \wedge^{\circ}, \square^{\bullet}, \square^{\circ}$ and store eagerly, obtaining leaves of the form $\Lambda\{\bot^{\circ}\}$.
ii. Apply any rule of nEK_c eagerly, until either finishing the proof with an axiom application or obtaining leaves of the form $\Lambda\{P^{\circ}\}$, where P is a positive formula in nEK_c, that is, having as main connective $\wedge$ or $\square$. Start again from step (i).

Axiom	Condition	First-Order Formula
$\mathsf{D}: \Box A \to \Diamond A$	Seriality	$\forall x \exists y. R(x, y)$
$\mathsf{T}: \Box A \to A \wedge A \to \Diamond A$	Reflexivity	$\forall x. R(x, x)$
$\mathsf{B}: A \to \Box\Diamond A \wedge \Diamond\Box A \to A$	Symmetry	$\forall x, y. R(x, y) \to R(y, x)$
$4: \Box A \to \Box\Box A \wedge \Diamond\Diamond A \to \Diamond A$	Transitivity	$\forall x, y, z.(R(x, y) \wedge R(y, z)) \to R(x, z)$
$5: \Box A \to \Box\Diamond A \wedge \Diamond\Box A \to \Diamond A$	Euclideaness	$\forall x, y, z.(R(x, y) \wedge R(x, z)) \to R(y, z)$

Table 1. Axioms and corresponding first-order conditions on R.

Observe that weakening is never applied, since a positive classical formula P° is totally decomposable into negative subformulas of the form N°, which are stored in the classical input context as $N^\triangledown$, or $\bot^\circ$. □

This result clarifies the role of weakening in nEK: it serves as a bridge between intuitionistic and classical parts of a derivation and its application can be restricted to just below classical rules.

Since weakening is not present, nEK_i matches exactly the system NIK in [Str13].

Fact 6 *The intuitionistic fragment of* nEK *is Straßburger's system* NIK*.*

For the classical fragment, the discipline presented in the proof of Proposition 1 is interesting *per se*, since it induces a focused flavor to nEK_c: neutral connectives are handled in the stoup, while rules on classical connectives are applied in classical context.

But this discipline does not match the focusing defined in [CMS16], since in that work the diamond is considered positive and the box negative, while the ecumenical system forces the opposite polarity assignment. For providing a fully focused system, we should adopt a *fully polarized syntax*, with polarized versions of conjunction and disjunction, as done *e.g.* in [LM11,CMS16]. This will be left for a future work.

7 Extensions

Depending on the application, several further modal logics can be defined as extensions of EK by simply restricting the class of frames we consider or, equivalently, by adding axioms over modalities. Many of the restrictions one can be interested in are definable as formulas of first-order logic, where the binary predicate $R(x, y)$ refers to the corresponding accessibility relation. Table 1 summarizes some of the most common logics, the corresponding frame property, together with the modal axiom capturing it [Sah75].

Since the intuitionistic fragment of nEK coincides with NIK, intuitionistic versions for the rules for the axioms d, t, b, 4, and 5 match the rules ($^\bullet$) and ($^\circ$) presented in [Str13], and are depicted in Figure 5, Appendix A.

For completing the ecumenical view, the classical ($^\triangledown$) rules for extensions are justified via translation to the labeled system labEK. For example, the labeled derivation on the left justifies the classical rule in the middle.

$$\dfrac{\dfrac{\dfrac{\dfrac{xRx, \mathcal{R}, \Sigma, x : \neg A \Rightarrow x : \bot}{xRx, \mathcal{R}, \Sigma, x : \Box\neg A \Rightarrow x : \bot}\ \Box L}{\mathcal{R}, \Sigma, x : \Box\neg A \Rightarrow x : \bot}\ \mathsf{T}}{\mathcal{R}, \Sigma \Rightarrow x : \Diamond_c A}\ \Diamond_c R}{\mathcal{R}, \Sigma, x : \neg\Diamond_c A \Rightarrow z : \bot}\ \neg L \qquad \dfrac{\Gamma^{\bot^\circ}\{A^\triangledown\}}{\Gamma^{\bot^\circ}\{\Diamond_c A^\triangledown\}}\ \mathsf{t}^\triangledown \qquad \dfrac{xRx, \Gamma \vdash z : C}{\Gamma \vdash z : C}\ \mathsf{T}$$

The rule T above right is the labeled rule corresponding to the axiom t [Sim94]. The rules $\mathsf{d}^\triangledown, \mathsf{b}^\triangledown, \mathsf{4}^\triangledown$ and $\mathsf{5}^\triangledown$, shown in Figure 5, are obtained in the same manner. In this way, we have proposed ecumenical modal systems for all the logics in the $\mathsf{S5}$ modal cube [BRV01].

8 Conclusion and future work

In this paper, we have presented a pure, nested proof system nEK for the ecumenical modal logic EK, together with pure fragments, and the modal cube extensions. We proved soundness of nEK w.r.t. the ecumenical birelational Kripke model via a translation to the labeled ecumenical modal system labEK. For completeness, we used the fact that EK axioms are provable in nEK, and we proved cut-elimination for nEK. Finally, having an ecumenical nested system allowed for extracting well known systems as fragments.

First of all, it should be noted that combining classical and intuitionistic modalities in the same logical *internal* system, which is a conservative extension of both, is not trivial. In fact, the labeled system presented in [MPPS20] makes an extensive use of negations in order to keep classical information *persistent*. This can be mimicked by adding an extra classical context for storing negative formulas, as done in Girard's classical system LC [Gir91], thus solving the "purity" part. On the other hand, there seems to be no trivial solution for removing labels from intuitionistic modal sequent systems where the distributivity of the diamond w.r.t. the disjunction holds. The solution adopted here was to extend the framework to nested sequents, where the tree structure describes the corresponding path in the Kripke structure, and labels can be eliminated.

It turns out that this mix of extra context, polarities, and nestings can be implosive, in the sense that adding a cut rule may lead to a collapse of the system to classical modal logic. For controlling the implosion, the cut rules should have a restrict use of polarities which, in turn, makes the cut-elimination proof non trivial.

There are many interesting ideas that can be proposed for the systems, axioms, and semantics, as indicated throughout the text, and many lines to be pursued in this research direction. First of all, we have proposed a proof discipline for nEK, which does not correspond to focusing in the modal systems presented in [CMS16]. In fact, the presence of weakening breaks down focusing, and we should investigate alternative ways of having a fully focused system. Moreover, it should be really interesting to study *typing* in ecumenical systems, obtaining as fragments well known typed modal systems. Finally, we plan to implement ecumenical provers, as well as to automate the cut-elimination proof in the L-Framework [OPR21].

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A Proof systems

$$\frac{}{x : p_i, \Gamma \Rightarrow x : p_i}\ \mathsf{init} \qquad \frac{}{x : \bot, \Gamma \Rightarrow z : C}\ \bot_L \qquad \frac{\Gamma \Rightarrow y : \bot}{\Gamma \Rightarrow x : A}\ \mathsf{W}$$

$$\frac{x : p_i, \Gamma \Rightarrow z : \bot}{x : p_c, \Gamma \Rightarrow z : \bot}\ L_c \qquad \frac{x : p_i, \Gamma \Rightarrow x : \bot}{\Gamma \Rightarrow x : p_c}\ R_c$$

$$\frac{x : A, x : B, \Gamma \Rightarrow z : C}{x : A \wedge B, \Gamma \Rightarrow z : C}\ \wedge L \qquad \frac{\Gamma \Rightarrow x : A \quad \Gamma \Rightarrow x : B}{\Gamma \Rightarrow x : A \wedge B}\ \wedge R \qquad \frac{x : \neg A, \Gamma \Rightarrow z : A}{x : \neg A, \Gamma \Rightarrow z : \bot}\ \neg L$$

$$\frac{x : A, \Gamma \Rightarrow x : \bot}{\Gamma \Rightarrow x : \neg A}\ \neg R \qquad \frac{x : A, \Gamma \Rightarrow z : C \quad x : B, \Gamma \Rightarrow z : C}{x : A \vee_i B, \Gamma \Rightarrow z : C}\ \vee_i L \qquad \frac{\Gamma \Rightarrow x : A_j}{\Gamma \Rightarrow x : A_1 \vee_i A_2}\ \vee_i R_j$$

$$\frac{x : A, \Gamma \Rightarrow z : \bot \quad x : B, \Gamma \Rightarrow z : \bot}{x : A \vee_c B, \Gamma \Rightarrow z : \bot}\ \vee_c L \qquad \frac{\Gamma, x : \neg A, x : \neg B \Rightarrow x : \bot}{\Gamma \Rightarrow x : A \vee_c B}\ \vee_c R$$

$$\frac{x : A \rightarrow_i B, \Gamma \Rightarrow x : A \quad x : B, \Gamma \Rightarrow z : C}{x : A \rightarrow_i B, \Gamma \Rightarrow z : C}\ \rightarrow_i L \qquad \frac{x : A, \Gamma \Rightarrow x : B}{\Gamma \Rightarrow x : A \rightarrow_i B}\ \rightarrow_i R$$

$$\frac{x : A \rightarrow_c B, \Gamma \Rightarrow x : A \quad x : B, \Gamma \Rightarrow z : \bot}{x : A \rightarrow_c B, \Gamma \Rightarrow z : \bot}\ \rightarrow_c L \qquad \frac{x : A, x : \neg B, \Gamma \Rightarrow x : \bot}{\Gamma \Rightarrow x : A \rightarrow_c B}\ \rightarrow_c R$$

$$\frac{xRy, y : A, x : \Box A, \Gamma \Rightarrow z : C}{xRy, x : \Box A, \Gamma \Rightarrow z : C}\ \Box L \qquad \frac{xRy, \Gamma \Rightarrow y : A}{\Gamma \Rightarrow x : \Box A}\ \Box R \qquad \frac{xRy, y : A, \Gamma \Rightarrow z : C}{x : \Diamond_i A, \Gamma \Rightarrow z : C}\ \Diamond_i L$$

$$\frac{xRy, \Gamma \Rightarrow y : A}{xRy, \Gamma \Rightarrow x : \Diamond_i A}\ \Diamond_i R \qquad \frac{xRy, y : A, \Gamma \Rightarrow z : \bot}{x : \Diamond_c A, \Gamma \Rightarrow z : \bot}\ \Diamond_c L \qquad \frac{x : \Box\neg A, \Gamma \Rightarrow x : \bot}{\Gamma \Rightarrow x : \Diamond_c A}\ \Diamond_c R$$

Fig. 1. Ecumenical modal system labEK. In rules $\Box R, \Diamond_i L, \Diamond_c L$, the eigenvariable y does not occur free in any formula of the conclusion. In the rule W, either $A \neq \bot$ or $x \neq y$.

$$\frac{}{\Lambda\{p_i^\bullet, p_i^\circ\}}\ \mathsf{init} \qquad \frac{}{\Gamma\{\bot^\bullet\}}\ \bot^\bullet \qquad \frac{\Gamma^{\bot^\circ}}{\Gamma}\ \mathsf{W} \qquad \frac{\Gamma^{\bot^\circ}\{p_i^\bullet\}}{\Gamma^{\bot^\circ}\{p_c^\bullet\}}\ p_c^\bullet \qquad \frac{\Gamma^{\bot^\circ}\{p_i^\triangledown\}}{\Gamma^{\bot^\circ}\{p_c^\triangledown\}}\ p_c^\triangledown$$

$$\frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}}\ \wedge^\bullet \qquad \frac{\Lambda\{A^\circ\} \quad \Lambda\{B^\circ\}}{\Lambda\{A \wedge B^\circ\}}\ \wedge^\circ \qquad \frac{\Gamma^*\{\neg A^\bullet, A^\circ\}}{\Gamma^{\bot^\circ}\{\neg A^\bullet\}}\ \neg^\bullet \qquad \frac{\Gamma^{\bot^\circ}\{A^\bullet\}}{\Gamma^{\bot^\circ}\{\neg A^\triangledown\}}\ \neg^\triangledown$$

$$\frac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \vee_i B^\bullet\}}\ \vee_i^\bullet \qquad \frac{\Lambda\{A_j^\circ\}}{\Lambda\{A_1 \vee_i A_2^\circ\}}\ \vee_{i_j}^\circ \qquad \frac{\Gamma^{\bot^\circ}\{A^\bullet\} \quad \Gamma^{\bot^\circ}\{B^\bullet\}}{\Gamma^{\bot^\circ}\{A \vee_c B^\bullet\}}\ \vee_c^\bullet \qquad \frac{\Gamma^{\bot^\circ}\{A^\triangledown, B^\triangledown\}}{\Gamma^{\bot^\circ}\{A \vee_c B^\triangledown\}}\ \vee_c^\triangledown$$

$$\frac{\Gamma^*\{A \rightarrow_i B^\bullet, A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \rightarrow_i B^\bullet\}}\ \rightarrow_i^\bullet \qquad \frac{\Lambda\{A^\bullet, B^\circ\}}{\Lambda\{A \rightarrow_i B^\circ\}}\ \rightarrow_i^\circ \qquad \frac{\Gamma^*\{A \rightarrow_c B^\bullet, A^\circ\} \quad \Gamma^{\bot^\circ}\{B^\bullet\}}{\Gamma^{\bot^\circ}\{A \rightarrow_c B^\bullet\}}\ \rightarrow_c^\bullet$$

$$\frac{\Gamma^{\bot^\circ}\{A^\bullet, B^\triangledown\}}{\Gamma^{\bot^\circ}\{A \rightarrow_c B^\triangledown\}}\ \rightarrow_c^\triangledown \qquad \frac{\Delta_1\{\Box A^\bullet, [A^\bullet, \Delta_2]\}}{\Delta_1\{\Box A^\bullet, [\Delta_2]\}}\ \Box^\bullet \qquad \frac{\Lambda\{[A^\circ]\}}{\Lambda\{\Box A^\circ\}}\ \Box^\circ \qquad \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond_i A^\bullet\}}\ \Diamond_i^\bullet \qquad \frac{\Lambda_1\{[A^\circ, \Lambda_2]\}}{\Lambda_1\{\Diamond_i A^\circ, [\Lambda_2]\}}\ \Diamond_i^\circ$$

$$\frac{\Gamma^{\bot^\circ}\{[A^\bullet]\}}{\Gamma^{\bot^\circ}\{\Diamond_c A^\bullet\}}\ \Diamond_c^\bullet \qquad \frac{\Delta_1^{\bot^\circ}\{\Diamond_c A^\triangledown, [A^\triangledown, \Delta_2^{\bot^\circ}]\}}{\Delta_1^{\bot^\circ}\{\Diamond_c A^\triangledown, [\Delta_2^{\bot^\circ}]\}}\ \Diamond_c^\triangledown \qquad \frac{\Gamma^*\{P^\triangledown, P^\circ\}}{\Gamma^{\bot^\circ}\{P^\triangledown\}}\ \mathsf{D} \qquad \frac{\Lambda\{N^\triangledown, \bot^\circ\}}{\Lambda\{N^\circ\}}\ \mathsf{store}$$

Fig. 2. Nested ecumenical modal system nEK. P is a *positive* formula, N is a *negative* formula.

$$\frac{}{\Lambda\{p_i^\bullet, p_i^\circ\}}\ \mathsf{init} \qquad \frac{}{\Gamma\{\bot^\bullet\}}\ \bot^\bullet \qquad \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}}\ \wedge^\bullet \qquad \frac{\Lambda\{A^\circ\} \quad \Lambda\{B^\circ\}}{\Lambda\{A \wedge B^\circ\}}\ \wedge^\circ$$

$$\frac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \vee_i B^\bullet\}}\ \vee_i^\bullet \qquad \frac{\Lambda\{A_j^\circ\}}{\Lambda\{A_1 \vee_i A_2^\circ\}}\ \vee_{i_j}^\circ \qquad \frac{\Gamma^*\{A \to_i B^\bullet, A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \to_i B^\bullet\}}\ \to_i^\bullet \qquad \frac{\Lambda\{A^\bullet, B^\circ\}}{\Lambda\{A \to_i B^\circ\}}\ \to_i^\circ$$

$$\frac{\Delta_1\{\Box A^\bullet, [A^\bullet, \Delta_2]\}}{\Delta_1\{\Box A^\bullet, [\Delta_2]\}}\ \Box^\bullet \qquad \frac{\Lambda\{[A^\circ]\}}{\Lambda\{\Box A^\circ\}}\ \Box^\circ \qquad \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond_i A^\bullet\}}\ \Diamond_i^\bullet \qquad \frac{\Lambda_1\{[A^\circ, \Lambda_2]\}}{\Lambda_1\{\Diamond_i A^\circ, [\Lambda_2]\}}\ \Diamond_i^\circ$$

Fig. 3. Intuitionistic fragment nEK_i.

$$\frac{}{\Gamma\{p_c^\bullet, p_c^\triangledown\}}\ \mathsf{init} \qquad \frac{}{\Gamma\{\bot^\bullet\}}\ \bot^\bullet \qquad \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}}\ \wedge^\bullet \qquad \frac{\Lambda\{A^\circ\} \quad \Lambda\{B^\circ\}}{\Lambda\{A \wedge B^\circ\}}\ \wedge^\circ$$

$$\frac{\Gamma^{\bot^\circ}\{A^\bullet\} \quad \Gamma^{\bot^\circ}\{B^\bullet\}}{\Gamma^{\bot^\circ}\{A \vee_c B^\bullet\}}\ \vee_c^\bullet \qquad \frac{\Gamma^{\bot^\circ}\{A^\triangledown, B^\triangledown\}}{\Gamma^{\bot^\circ}\{A \vee_c B^\triangledown\}}\ \vee_c^\triangledown \qquad \frac{\Gamma^*\{A \to_c B^\bullet, A^\circ\} \quad \Gamma^{\bot^\circ}\{B^\bullet\}}{\Gamma^{\bot^\circ}\{A \to_c B^\bullet\}}\ \to_c^\bullet$$

$$\frac{\Gamma^{\bot^\circ}\{A^\bullet, B^\triangledown\}}{\Gamma^{\bot^\circ}\{A \to_c B^\triangledown\}}\ \to_c^\triangledown \qquad \frac{\Delta_1\{\Box A^\bullet, [A^\bullet, \Delta_2]\}}{\Delta_1\{\Box A^\bullet, [\Delta_2]\}}\ \Box^\bullet \qquad \frac{\Lambda\{[A^\circ]\}}{\Lambda\{\Box A^\circ\}}\ \Box^\circ$$

$$\frac{\Gamma^{\bot^\circ}\{[A^\bullet]\}}{\Gamma^{\bot^\circ}\{\Diamond_c A^\bullet\}}\ \Diamond_c^\bullet \qquad \frac{\Delta_1^{\bot^\circ}\{\Diamond_c A^\triangledown, [A^\triangledown, \Delta_2^{\bot^\circ}]\}}{\Delta_1^{\bot^\circ}\{\Diamond_c A^\triangledown, [\Delta_2^{\bot^\circ}]\}}\ \Diamond_c^\triangledown \qquad \frac{\Gamma^*\{P^\triangledown, P^\circ\}}{\Gamma^{\bot^\circ}\{P^\triangledown\}}\ \mathsf{D} \qquad \frac{\Lambda\{N^\triangledown, \bot^\circ\}}{\Lambda\{N^\circ\}}\ \mathsf{store}$$

Fig. 4. Classical fragment nEK_c.

$$\frac{\Gamma\{\Box A^\bullet, [A^\bullet]\}}{\Gamma\{\Box A^\bullet\}}\ \mathsf{d}^\bullet \qquad \frac{\Lambda\{[A^\circ]\}}{\Lambda\{\Diamond_i A^\circ\}}\ \mathsf{d}^\circ \qquad \frac{\Gamma^{\bot^\circ}\{[A^\triangledown]\}}{\Gamma^{\bot^\circ}\{\Diamond_c A^\triangledown\}}\ \mathsf{d}^\triangledown \qquad \frac{\Gamma\{\Box A^\bullet, A^\bullet\}}{\Gamma\{\Box A^\bullet\}}\ \mathsf{t}^\bullet \qquad \frac{\Lambda\{A^\circ\}}{\Lambda\{\Diamond_i A^\circ\}}\ \mathsf{t}^\circ \qquad \frac{\Gamma^{\bot^\circ}\{A^\triangledown\}}{\Gamma^{\bot^\circ}\{\Diamond_c A^\triangledown\}}\ \mathsf{t}^\triangledown$$

$$\frac{\Delta_1\{[\Delta_2, \Box A^\bullet], A^\bullet\}}{\Delta_1\{[\Delta_2, \Box A^\bullet]\}}\ \mathsf{b}^\bullet \qquad \frac{\Lambda_1\{[\Lambda_2], A^\circ\}}{\Lambda_1\{[\Lambda_2, \Diamond_i A^\circ]\}}\ \mathsf{b}^\circ \qquad \frac{\Delta_1^{\bot^\circ}\{[\Delta_2^{\bot^\circ}], A^\triangledown\}}{\Delta_1^{\bot^\circ}\{[\Delta_2^{\bot^\circ}, \Diamond_c A^\triangledown]\}}\ \mathsf{b}^\triangledown \qquad \frac{\Delta_1\{[\Delta_2, \Box A^\bullet], \Box A^\bullet\}}{\Delta_1\{[\Delta_2], \Box A^\bullet\}}\ \mathsf{4}^\bullet$$

$$\frac{\Lambda_1\{[\Lambda_2, \Diamond_i A^\circ]\}}{\Lambda_1\{[\Lambda_2], \Diamond_i A^\circ\}}\ \mathsf{4}^\circ \qquad \frac{\Delta_1^{\bot^\circ}\{[\Delta_2^{\bot^\circ}, \Diamond_i A^\circ]\}}{\Delta_1^{\bot^\circ}\{[\Delta_2^{\bot^\circ}], \Diamond_c A^\triangledown\}}\ \mathsf{4}^\triangledown \qquad \frac{\Gamma\{[\Box A^\bullet][\Box A^\bullet]\}}{\Gamma\{[\Box A^\bullet][\emptyset]\}}\ \mathsf{5}^\bullet \qquad \frac{\Lambda\{[\emptyset][\Diamond_i A^\circ]\}}{\Lambda\{[\Diamond_i A^\circ][\emptyset]\}}\ \mathsf{5}^\circ \qquad \frac{\Gamma^{\bot^\circ}\{[\emptyset][\Diamond_c A^\triangledown]\}}{\Gamma^{\bot^\circ}\{[\Diamond_c A^\triangledown][\emptyset]\}}\ \mathsf{5}^\triangledown$$

Fig. 5. Ecumenical modal extensions for axioms $\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}$ and $\mathsf{5}$.

B Proof of Theorem 3

The proof is by mutual induction, with inductive measure (n, m) where m is the cut-height and n is the ecumenical weight of the cut-formula, defined as

$$\begin{array}{ll}
\mathsf{ew}(P_i) = \mathsf{ew}(\bot) = 0 & \mathsf{ew}(A \star B) = \mathsf{ew}(A) + \mathsf{ew}(B) + 1 \text{ if } \star \in \{\wedge, \to_i, \vee_i\} \\
\mathsf{ew}(P_c) = 4 & \mathsf{ew}(\heartsuit A) = \mathsf{ew}(A) + 1 \text{ if } \heartsuit \in \{\neg, \Diamond_i, \Box\} \\
\mathsf{ew}(\Diamond_c A) = \mathsf{ew}(A) + 4 & \mathsf{ew}(A \circ B) = \mathsf{ew}(A) + \mathsf{ew}(B) + 4 \text{ if } \circ \in \{\to_c, \vee_c\}
\end{array}$$

Intuitively, the ecumenical weight measures the amount of extra information needed (the negations added) in order to define the classical connectives from the intuitionistic and neutral ones. We will sketch next the most important cut-reductions.

– Base cases. Consider the derivation below left

$$\dfrac{\overset{\pi}{\Lambda\left\{p_i^\circ\right\}} \qquad \dfrac{}{\Gamma\left\{p_i^\bullet\right\}}\;\mathit{init}}{\Lambda \otimes \Gamma\{\emptyset\}}\;\mathsf{cut}^\circ \qquad\qquad \dfrac{\overset{\pi}{\Lambda\left\{p_i^\circ\right\}}}{\Lambda \otimes \Gamma^*\left\{p_i^\circ\right\}}\;\mathsf{W}_c$$

If $p_i^\bullet$ is principal, then $\Gamma\left\{p_i^\bullet\right\} = \Gamma^*\left\{p_i^\circ, p_i^\bullet\right\}$ and the reduction is the one above right. If $p_i^\bullet$ is not principal, then $\{p_i^\circ, p_i^\bullet\} \in \Lambda \otimes \Gamma\{\emptyset\}$ and the reduction is the trivial one. Similar analyses hold for $\mathsf{cut}^\triangledown$, when the left premise is an instance of init, and for the other axioms.

– Non-principal cases. In all the cases where the cut-formula is not principal in one of the premises, the cut moves upwards. We illustrate the most significant case, where a decide rule is applied, as in the derivation below left.

$$\dfrac{\dfrac{\overset{\pi_1}{\Lambda\{N^\triangledown, P^\triangledown, P^\circ\}}}{\Lambda^{\perp^\circ}\{N^\triangledown, P^\triangledown\}}\;\mathsf{D} \qquad \overset{\pi_2}{\Gamma\{N^\bullet\}}}{\Lambda \otimes \Gamma\{P^\triangledown\}\{\emptyset\}}\;\mathsf{cut}^\triangledown \qquad\qquad \dfrac{\dfrac{\overset{\pi_1}{\Lambda\{N^\triangledown, P^\triangledown, P^\circ\}} \qquad \overset{\pi_2}{\Gamma^{\perp^\circ}\{N^\bullet\}}}{\Lambda \otimes \Gamma^*\{P^\triangledown, P^\circ\}\{\emptyset\}}\;\mathsf{cut}^\triangledown}{\Lambda \otimes \Gamma^{\perp^\circ}\{P^\triangledown\}\{\emptyset\}}\;\mathsf{D}$$

If $N^\bullet$ is principal in π_2, then $\Gamma = \Gamma^{\perp^\circ}$ and D moves downwards the cut, obtaining the derivation above right. Otherwise, the cut moves upwards in the right premise.

– Principal cases. If the cut formula is principal in both premises, then we need to be extra-careful with the polarities. We illustrate below the reduction for case where $N = A \rightarrow_c B$, with A, B positive.

$$\dfrac{\dfrac{\overset{\pi_1}{\Lambda^{\perp^\circ}\{A^\bullet, B^\triangledown\}}}{\Lambda^{\perp^\circ}\{A \rightarrow_c B^\triangledown\}}\;\rightarrow_c^\triangledown \qquad \dfrac{\overset{\pi_2}{\Gamma^*\{A \rightarrow_c B^\bullet, A^\circ\}} \qquad \overset{\pi_3}{\Gamma^{\perp^\circ}\{B^\bullet\}}}{\Gamma^{\perp^\circ}\{A \rightarrow_c B^\bullet\}}\;\rightarrow_c^\bullet}{\Lambda \otimes \Gamma^{\perp^\circ}\{\emptyset\}}\;\mathsf{cut}_0^\triangledown$$

reduces to

$$\dfrac{\dfrac{\dfrac{\overset{\pi_3}{\Gamma^{\perp^\circ}\{B^\bullet\}}}{\Gamma^{\perp^\circ}\{\neg B^\triangledown\}}\;\neg^\triangledown \qquad \dfrac{\dfrac{\dfrac{\overset{\pi_1}{\Lambda^{\perp^\circ}\{A^\bullet, B^\triangledown\}}}{\Lambda^{\perp^\circ}\{A \rightarrow_c B^\triangledown\}}\;\rightarrow_c^\triangledown \qquad \overset{\pi_2}{\Gamma^*\{A \rightarrow_c B^\bullet, A^\circ\}}}{\Lambda \otimes \Gamma^{\perp^\circ}\{A^\circ\}}\;\mathsf{cut}_2^\triangledown \qquad \overset{\pi_1^{\equiv}}{\Lambda^{\perp^\circ}\{A^\bullet, \neg B^\bullet\}}}{\Lambda^2 \otimes \Gamma^{\perp^\circ}\{\neg B^\bullet\}}\;\mathsf{cut}^\circ}{\Lambda^2 \otimes \Gamma^* \otimes \Gamma^{\perp^\circ}\{\emptyset\}}\;\mathsf{cut}_1^\triangledown}{\Lambda \otimes \Gamma^{\perp^\circ}\{\emptyset\}}\;\mathsf{C}_c$$

where $\pi_1^{\equiv}$ is the same as π_1 where every application of the rule decide D over $B^\triangledown$ is substituted by an application of $\neg^\bullet$ over $B^\bullet$. Observe that the cut-formula of $\mathsf{cut}_1^\triangledown$ has lower ecumenical weight than $\mathsf{cut}_0^\triangledown$, while the cut-height of $\mathsf{cut}_2^\triangledown$ is smaller than $\mathsf{cut}_0^\triangledown$. Finally, observe that this is a non-trivial cut-reduction: usually, the cut over the implication is replaced by a cut over B first. Due to polarities, if B is positive, then $\neg B$ is negative and cutting over it will add to the left context the classical information B, hence mimicking the behavior of formulas in the right input context.

C Alternative proof of soundness

In this section we will sketch an alternative proof of soundness of nEK w.r.t. the ecumenical birelational Kripke model. We start by interpreting nested sequents as ecumenical formulas.

Definition 6. *The formula translation* $\mathrm{et}(\cdot)$ *for ecumenical nested sequents is given by*

$$\begin{array}{rlrl} \mathrm{et}(\varnothing) &:= \top & \mathrm{et}(A^{\bullet}, \Lambda) &:= A \wedge \mathrm{et}(\Lambda) \\ \mathrm{et}(A^{\triangledown}, \Lambda) &:= \neg A \wedge \mathrm{et}(\Lambda) & \mathrm{et}([\Lambda_1], \Lambda_2) &:= \Diamond_i \mathrm{et}(\Lambda_1) \wedge \mathrm{et}(\Lambda_2) \\ \mathrm{et}(\Lambda, A^{\circ}) &:= \mathrm{et}(\Lambda) \to_i A & \mathrm{et}(\Lambda, [\Gamma]) &:= \mathrm{et}(\Lambda) \to_i \Box \mathrm{et}(\Gamma) \end{array}$$

where all occurrences of $A \wedge \top$ *and* $\top \to_i A$ *are simplified to* A. *We say a sequent is* valid *if its corresponding formula is valid.*

The following technical lemma holds in nEK, adapting the proof from NIK.

Lemma 2. *[Str13, Lemmas 4.3 and 4.4] Let* Δ *and* Σ *be input (resp. full) sequents, and* $\Gamma\{\,\}$ *be a full context (resp.* $\Lambda\{\,\}$ *be an input context). If* $\mathrm{et}(\Delta) \to_i \mathrm{et}(\Sigma)$ *is valid, then*

$$\mathrm{et}(\Gamma\{\Sigma\}) \to_i \mathrm{et}(\Gamma\{\Delta\}) \text{ is valid.} \qquad \mathrm{et}(\Lambda\{\Delta\}) \to_i \mathrm{et}(\Lambda\{\Sigma\}) \text{ is valid.}$$

The next theorem shows that the rules of nEK preserve validity in ecumenical modal frames w.r.t. the formula interpretation $\mathrm{et}(\cdot)$.

Theorem 7. *Let*

$$\frac{\Gamma_1 \quad \dots \quad \Gamma_n}{\Gamma}\, r \quad n \in \{0, 1, 2\}$$

be an instance of the rule r *in the system* nEK. *Then* $\mathrm{et}(\Gamma_1) \wedge \ldots \wedge \mathrm{et}(\Gamma_n) \to_i \mathrm{et}(\Gamma)$ *is valid in the birrelational ecumenical semantics.*

Proof. The proof for the intuitionistic propositional and modal connectives follows the same lines as in [Str13]. For the other cases, due to Lemma 2, it is sufficient to show that the following formulas are valid

1. for W: $\bot \to_i A$
2. for $\neg^{\bullet}$: $(\neg A \to_i A) \to_i (\neg\neg A)$
3. for $\neg^{\triangledown}$: $\neg A \to_i \neg A$
4. for $\vee_c^{\bullet}$: $(\neg A \wedge \neg B) \to_i (\neg(A \vee_c B))$
5. for $\vee_c^{\triangledown}$: $(\neg(\neg A \wedge \neg B)) \to_i (A \vee_c B)$
6. for $\to_c^{\bullet}$: $(((A \to_c B) \to_i A) \wedge (\neg B)) \to_i (\neg(A \to_c B))$
7. for $\to_c^{\triangledown}$: $(\neg(A \wedge \neg B)) \to_i (A \to_c B)$
8. for $p_c^{\bullet}$: $(\neg p_i) \to_i (\neg p_c)$
9. for $p_c^{\triangledown}$: $(\neg\neg p_i) \to_i p_c$
10. for $\Diamond_c^{\bullet}$: $(\neg \Diamond_i A) \to_i (\neg \Diamond_c A)$
11. for $\Diamond_c^{\triangledown}$: $(\Box \neg A) \to_i (\neg \Diamond_c A)$

But all such proofs are straightforward.